

Splitting Line Patterns in Free Groups

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ABSTRACT. There is a canonical cyclic JSJ-decomposition of a free group relative to a finite list of words in the group.

This provides a characterization of virtually geometric multiwords. They are the multiwords that are built from geometric pieces.

1. INTRODUCTION

Let $F = F_n$ be a free group of finite rank $n > 1$, and let $\underline{w} = \{w_1, \dots, w_k\} \subset F$ be a *multiword*, a finite list of nontrivial words. The goal of this paper is to find splittings of F *relative to* \underline{w} (rel \underline{w}), splittings of F as a free product with amalgamation over cyclic subgroups in such a way that each word of \underline{w} is conjugate into some factor.

First consider the problem in a topological setting. Consider the surface in Figure 1. The labeled curves are generators of the fundamental group $F_5 = \langle a, b, c, d, e \rangle$.

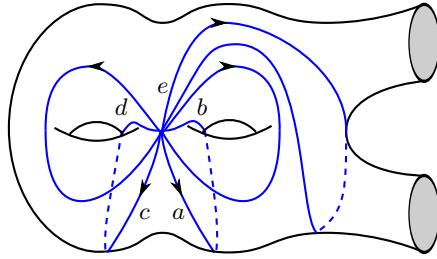


FIGURE 1. A surface with chosen generators of the fundamental group

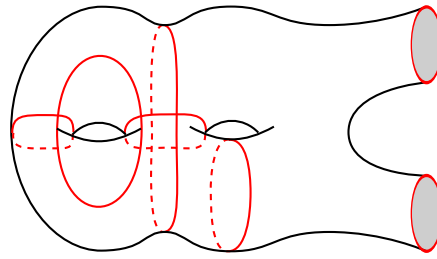


FIGURE 2. A multicurve realizing $\underline{w} = \{a, c, d, e, \bar{a}c, dc\bar{d}\bar{c}, dc\bar{d}\bar{c}ab\bar{a}\bar{b}\bar{e}\}$

Consider the multiword $\underline{w} = \{a, c, d, e, \bar{a}c, dc\bar{d}\bar{c}, dc\bar{d}\bar{c}ab\bar{a}\bar{b}\bar{e}\}$. Figure 2 shows a *multicurve* realizing \underline{w} . This is a collection of curves freely homotopic to curves representing the words of \underline{w} .

Splittings of surfaces as free products with cyclic amalgamation come from simple closed curves on the surface. Splitting relative to a multiword requires us to find a simple closed curve disjoint from the curves representing the words of the multiword.

In this example, two of the curves are the boundary curves of the surface. Take a small regular neighborhood of the remaining curves. There are two components;

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one is an annulus, and the other is more complicated. The complement consists of two discs, an annulus, and a sphere with four boundary components.

Cutting off discs does not give us any information, so consider the union of the discs with the more complicated surface. This subsurface is a torus with two boundary components, and it is *filled* by the multicurve; every essential simple closed curve in the subsurface intersects one of the curves of the multicurve. This subsurface does not split relative to the multiword. This filled subsurface is an example of what we will call a *rigid* vertex.

We also have two parallel annuli, one a neighborhood of a curve and one in the complement. Take their union to get an annulus with a curve of the multicurve as its core. This core curve gives us a splitting of F relative to the multiword.

The remaining complementary component is a sphere with four boundary components. It is *empty*, in the sense that there are no curves of the multicurve in the interior of this subsurface. It is possible to split this subsurface relative to the multiword, but not in a canonical way. Every simple closed curve would give us a splitting, but for each simple closed curve there is another simple closed curve intersecting it. The splittings from intersecting curves would be incompatible, so rather than make an arbitrary choice we keep the empty subsurface and remember that it splits in incompatible ways.

So far, we have the splitting from the annulus and splittings from the empty subsurface. There is one remaining. We can split along the boundary curve of the filled subsurface where it meets the empty subsurface. To make the picture symmetric, insert another annulus at the interface of the filled subsurface and the empty subsurface, as in Figure 3. Figure 4 shows the corresponding graph of groups decomposition of $F \text{ rel } \underline{w}$. It has two cyclic vertex groups corresponding to the two annuli and two rank 3 free vertex groups corresponding to the two higher complexity subsurfaces. The edge groups are all cyclic. The labels at the ends of each edge indicate the image of a generator of the edge group in the adjacent vertex group.

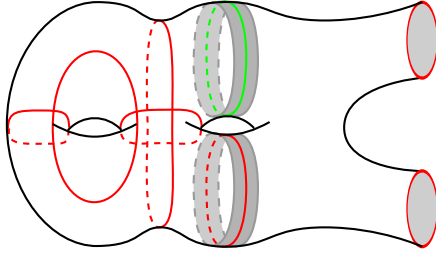


FIGURE
3. Decomposition
of a surface.

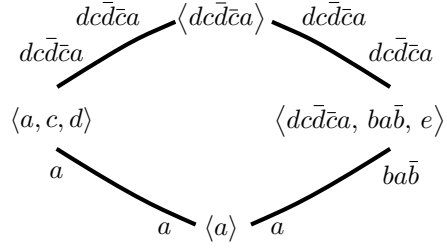


FIGURE
4. Corresponding
graph of groups

In summary, we have a decomposition of the original surface into subsurfaces. The decomposition is bipartite: there is a collection of annuli and a collection of more complicated subsurfaces, and each subsurface is adjacent only to members of the opposite collection. Among the more complicated subsurfaces there are those that are filled and those that are empty. Every splitting of the total surface relative to the multicurve comes from either the core curve of one of the annuli or a simple closed curve in the interior of one of the empty surface pieces.

We call this decomposition the *relative JSJ-Decomposition* ($rJSJ$). It is a decomposition that encodes all possible cyclic splittings relative to the given multiword. The main result of this paper is that given a multiword in a free group there is a canonical group-theoretic relative JSJ-Decomposition. In general it will not be the case that the free group corresponds to the fundamental group of a surface with boundary and the pieces of the $rJSJ$ correspond to subsurfaces. Nonetheless, the statement will be completely analogous to the simple example above:

Relative JSJ-Decomposition Theorem (Theorem 3.26). *Let \underline{w} be a multiword in a free group F such that F does not split freely relative to \underline{w} . There exists a canonical relative JSJ-decomposition ($rJSJ$), a graph of groups decomposition Γ of F relative to \underline{w} with cyclic edge groups, satisfying the following conditions:*

- (1) Γ satisfies certain normalization hypotheses.
- (2) Γ is universal. If F splits over a cyclic subgroup relative to \underline{w} then the cyclic subgroup is conjugate into one of the vertex groups.
- (3) Γ is maximal. It can not be refined and still satisfy these conditions.

Moreover, the $rJSJ$ is characterized by splitting F over the stabilizers of cut points and uncrossed cut pairs in a certain topological space $\mathbf{D}_{\underline{w}}$. There are three mutually exclusive possibilities:

- (a) (F, \underline{w}) is rigid. $\mathbf{D}_{\underline{w}}$ has no cut points or cut pairs. The $rJSJ$ is trivial, a single vertex stabilized by F .
- (b) (F, \underline{w}) is a QH-surface. $\mathbf{D}_{\underline{w}}$ is a circle. The $rJSJ$ is trivial, a single vertex stabilized by F .
- (c) The $rJSJ$ is nontrivial. For every non-cyclic vertex group G the pair $(G, \text{Ind}_{\Gamma}^G(\underline{w}))$ is either rigid or a QH-surface.

Consequently, if F splits over a cyclic subgroup relative to \underline{w} then the cyclic subgroup is conjugate into one of the cyclic vertices or one of the QH-surface vertices of the $rJSJ$.

We will say that (F, \underline{w}) is *rigid* if the corresponding decomposition space $\mathbf{D}_{\underline{w}}$ is connected without cut points or cut pairs. The fact that such a line pattern is quasi-isometrically rigid is the main result of [7]. Rigidity implies that this vertex does not split relative to the multiword (Corollary 3.2).

Statement (b) means that F is the fundamental group of a surface with boundary components and \underline{w} represents the boundary curves.

In the setting of the example, statement (c) means that G is the fundamental group of a subsurface, and $\text{Ind}_{\Gamma}^G(\underline{w})$ is the multiword in G consisting of the words of \underline{w} conjugate into G as well as any boundary curves of the subsurface that are not boundary curves of the total surface.

Remarks. In general one does not have a canonical JSJ-decomposition of a group. Rather, there is a deformation space of JSJ-decompositions [10]. The normalization hypotheses of condition (1) pick out a particular point of the deformation space.

It is usual in the various JSJ theories to call the vertex groups “rigid” and “flexible”. The term “quadratically hanging surface” is from Sela’s JSJ theory [18]. In more general versions of JSJ theory the flexible vertex groups are often certain fiber bundles over quadratically hanging 2-orbifolds [10]. Since we are working in free groups we will not need this generality; QH-surfaces are all that will occur as flexible vertices.

A “rigid” vertex is usually defined to be one that does not split further. This leads to the annoying possibility that “rigid” and “flexible” may not be mutually exclusive. The example that is relevant in this paper is the sphere with three boundary components. We will use a definition of “rigid” that excludes this example, so “rigid” and “QH-surface” will be mutually exclusive.

In the surface example above there is a subtle point. One splitting came from a boundary curve of the filled subsurface. This curve was not a member of the multicurve, but, in a sense, it might just as well have been; the rJSJ would not have been any different if it had been included in the first place. Characterizing such “missing” curves in the case of the free group is not so trivial. In fact, it is our main task. The rest of the theorem follows rather easily.

1.1. Line Patterns and Decomposition Spaces. Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a free basis for F , and let \underline{w} be a multiword in F . Think of the words of \underline{w} as closed paths in the rose with n edges labeled by the b_i . The universal cover of the rose is a tree \mathcal{T} that is a Cayley graph of F . Define the *line pattern* $\mathcal{L}_{\underline{w}}$ generated by \underline{w} to be the collection of coarse equivalence classes of lifts of the closed paths in the rose coming from \underline{w} . Each individual lift picks out two points on the boundary at infinity of \mathcal{T} . The convex hull of these two points is a geodesic of \mathcal{T} coarsely equivalent to the lifted curve. These are the “lines” that make up the line pattern.

Changing a multiword by conjugating an element or replacing an element with a root or a power does not change the associated line pattern. In other words, we are really interested in finite collections of conjugacy classes of maximal cyclic subgroups. We will implicitly treat multiwords \underline{w} and \underline{w}' as equivalent if the words in \underline{w} belong to the same collection of conjugacy classes of maximal cyclic subgroups as the words of \underline{w}' . We will also say $g \in \underline{w}$ if the conjugacy class of the maximal cyclic subgroup containing g is in \underline{w} . With this notion of equivalence, a line pattern determines a generating multiword.

There is a topological space, the *decomposition space*, \mathbf{D} (or $\mathbf{D}_{\mathcal{L}}$ or $\mathbf{D}_{\underline{w}}$ if we want to be explicit) associated to a line pattern. It is obtained as a quotient of the boundary at infinity of the tree \mathcal{T} by identifying the two endpoints of each line $l \in \mathcal{L}$.

Any quasi-isometry of F that coarsely preserves the line pattern extends to a homeomorphism of the decomposition space. In particular, F acts on \mathbf{D} by homeomorphisms, and \mathbf{D} is independent of the choice of basis of F . We will use the topology of the decomposition space to construct a simplicial tree with an F -action. This will give us the rJSJ.

Several facts about the relationship between relative splittings and the topology of the decomposition space are already known:

- (1) F splits freely rel \underline{w} if and only if \mathbf{D} is not connected. (Corollary 2.4)
- (2) Any cut point of \mathbf{D} is stabilized by a conjugate of some word w of \underline{w} , and F splits over $\langle w \rangle$ rel \underline{w} . (Lemma 2.7 and Proposition 3.11)
- (3) If F splits over $\langle g \rangle$ rel \underline{w} then the image of $(g^\infty, \bar{g}^\infty)$ in \mathbf{D} is either a cut point or a cut pair. It is a cut point if and only if $g \in \underline{w}$. (Lemma 3.1 and Lemma 2.7)
- (4) (F, \underline{w}) is a QH-surface if and only if \mathbf{D} is a circle. (Lemma 2.11)

These observations can be attributed to Otal [15]. He proves in addition that if the decomposition space is planar and each element of the multiword is full width, then the presence of cut pairs implies the existence of a splitting rel \underline{w} .

In Theorem 3.12 we remove the additional hypotheses: If (F, \underline{w}) is not rigid then either F splits rel \underline{w} or (F, \underline{w}) is a sphere with three boundary components.

To do this we characterize the “missing” curves mentioned earlier. In a circle every pair of points is a cut pair, and for each cut pair there is another whose two points lie in opposite complementary components of the first, and vice versa. We say that such cut pairs are *crossing*.

In the decomposition space it may be that there are cut pairs that are not crossed by any other cut pairs. We call these *uncrossed*. We will show that every cut pair in the decomposition space either is uncrossed or belongs to a circle, in a certain sense. Proposition 3.7 shows that an uncrossed cut pair is rational, it is stabilized by a cyclic subgroup $\langle g \rangle < F$. Once we know this it is not hard to see Otal’s splitting argument still applies, so F splits over $\langle g \rangle$ rel \underline{w} . This subgroup $\langle g \rangle$ is the analogue of the “missing” boundary curve of the filled subsurface.

We build a simplicial tree from the collection of cut points and uncrossed cut pairs of \mathbf{D} . The F action on \mathbf{D} gives us a simplicial cocompact F action on this tree. The quotient graph of groups is the rJSJ. The result is canonical, since the tree is determined by the topology of \mathbf{D} .

1.2. Virtually Geometric Multiwords. In Section 4 we will apply the Relative JSJ-Decomposition Theorem to characterize virtually geometric multiwords.

A multiword in F is said to be *geometric* if it can be represented by an embedded multicurve in the boundary of a handlebody with fundamental group F . A multiword is *virtually geometric* if it becomes geometric upon lifting to a finite index subgroup of F .

Otal’s main result in [15], suitable reinterpreted, is that in the case that the rJSJ is trivial the multiword is geometric if and only if the decomposition space is planar. Furthermore, planarity of the decomposition space can be deduced from the Whitehead graph of the multiword.

The Relative JSJ-Decomposition Theorem provides a reduction of the virtual geometricity question:

Characterization of Virtual Geometricity (Theorem 4.10). *For a multiword in a free group, the following are equivalent:*

- (1) *The multiword is virtually geometric.*
- (2) *The decomposition space is planar.*
- (3) *For every non-cyclic vertex group of the rJSJ, the induced multiword is geometric.*

Thus, virtually geometric multiwords are exactly those that are built from geometric pieces.

1.3. Relation to Other People’s Work.

1.3.1. Boundaries of Hyperbolic Groups. It follows from Stallings’s Theorem [19] that the boundary of a hyperbolic group is disconnected if and only if the group splits over a finite group.

Bowditch [4] shows that if the boundary is connected, and if the group is not a “virtual semitriangle group”, then splittings over virtually cyclic groups come from the cut pair structure of the boundary.

The results of this paper say that the analogous statements are true for relative splittings of free groups with the decomposition space replacing the boundary of the hyperbolic group.

Proposition 3.4 is a generalization of a result of Bowditch that determines when collections of cut pairs form a circle. Among the technical differences are the facts that for hyperbolic boundaries there are no global cut points and every local cut point belongs to a cut pair. Neither of these are true for our decomposition spaces.

1.3.2. The Relatively Hyperbolic Perspective. In the construction of the decomposition space we take an element w of the free group and identify the two endpoints of its axis. Essentially we have declared that a hyperbolic element now acts parabolically. One can make this precise by saying that the free group is hyperbolic relative to the cyclic groups generated by elements of the multiword. It can be shown that the decomposition space is precisely Bowditch’s boundary of a relatively hyperbolic group [6]. Bowditch [5] extends the ideas used in splitting hyperbolic groups according to the topology of the boundary to show that the cut point structure of the relatively hyperbolic boundary gives a canonical “peripheral splitting”, a splitting of the relatively hyperbolic group over the parabolic subgroups such that parabolic subgroups are conjugate into the factors.

Bowditch’s results specialized to free groups hyperbolic relative to a multiword give the same graph of groups as Otal’s argument applied to the cut points of the decomposition space. However, this does not capture all relative splittings. It does not recognize the “missing curve” in the introduction, for example.

1.3.3. Cactus Trees. Since this paper was first written, a paper of Papasoglu and Swenson [16] about the “cactus tree” of a metric space has appeared. They prove a very general result about encoding the minimal cut sets of a continuum in a tree, again, generalizing Bowditch’s result for boundaries of hyperbolic groups. In our setting their theorem says that if the decomposition space is connected with no cut points then the set of cut pairs can be encoded in a simplicial tree. An alternate approach to producing the rJSJ would then be to apply results of Otal or Bowditch to produce a splitting from the cut points of the decomposition space, and then apply Papasoglu and Swenson’s result to produce splittings of the vertex groups from their cut pairs, refining the original decomposition.

We would still have to do the work of determining the properties of the rJSJ. Also, philosophically, the uncrossed cut pairs are “just as good as” cut points. The missing curve in the surface example really ought to have been there, and the same splitting argument works just as well when they are included.

1.4. Plan of the Paper. In Section 2 we fix definitions and recall some results about Whitehead graphs and decomposition spaces. This machinery was developed by the author and Macura in a previous paper [7]. The reader is referred to that paper for proofs and more details.

The Relative JSJ-Decomposition Theorem is proven in Section 3. In Section 3.1 we generalize a result of Bowditch that tells us when the decomposition space has cut points or uncrossed cut pairs. In Section 3.2 we show that cut points and uncrossed cut pairs have cyclic stabilizers and that there are finitely many orbits of

them. In Section 3.3 we generalize a result of Otal that shows that we can use any equivariant collection of cut points and uncrossed cut pairs in the decomposition space to produce a graph of groups decomposition of F rel \underline{w} . In Section 3.4 we examine the structure of the decomposition space in comparison to a graph of groups splitting and determine when a graph of groups decomposition can be refined. Finally, in Section 3.5 we put these pieces together to prove the Relative JSJ-Decomposition Theorem.

Section 4 applies the Relative JSJ-Decomposition Theorem to prove the Characterization of Virtual Geometricity.

I would like to thank Jason Manning for bringing Otal's paper to my attention and for helpful comments on an earlier version of this paper.

2. PRELIMINARIES

2.1. Free Groups, Line Patterns and Decomposition Spaces. Let $F = F_n$ be a free group of rank $n > 1$. For $g \in F$, let \bar{g} denote inverse of g .

A nontrivial element $g \in F$ is *indivisible* if it is not a proper power of another element. Equivalently, the cyclic subgroup $\langle g \rangle$ is maximal; it is not properly contained in a cyclic subgroup of F .

The *width* of g is the rank of the smallest free factor of F containing g . An element is *full width* if its width is equal to the rank of F . Full width is also known as *diskbusting*, particularly in the context of 3-manifolds.

A *basis* or *free basis* or *free generating set* of F is a generating set consisting of exactly n elements $\mathcal{B} = \{b_1, \dots, b_n\}$.

A subset $\{g_1, \dots, g_k\}$ is *basic* if it is a subset of a some basis. In particular, a single element g is *basic* if and only if it is indivisible and width one. The term *primitive* is also commonly used for our basic elements.

The *degree* of a homomorphism from the integers into a free group is the index of the image in the maximal cyclic subgroup containing the image.

The Cayley graph of F with respect to a basis \mathcal{B} is a tree \mathcal{T} whose vertices are in bijection with elements of F . There is an edge from vertex g to vertex h if and only if there exists a $b_i \in \mathcal{B}$ such that $gb_i = h$. We make this a metric space by assigning each edge length one; F acts isometrically on \mathcal{T} by left multiplication.

The tree \mathcal{T} has a boundary at infinity $\partial\mathcal{T}$ that is homeomorphic to a Cantor set. This compactifies the tree, $\overline{\mathcal{T}} = \mathcal{T} \cup \partial\mathcal{T}$ is a compact topological space whose topology on \mathcal{T} agrees with the metric topology.

Two subsets of \mathcal{T} are *coarsely equivalent*, written $\stackrel{c}{=}$, if they have bounded Hausdorff distance.

For any two points $\xi, \xi' \in \overline{\mathcal{T}}$ there exists a unique geodesic $[\xi, \xi']$ joining them.

For a nontrivial $h \in F$, we denote by gh^∞ the point in $\partial\mathcal{T}$ that is the limit of the sequence of vertices (gh^i) in \mathcal{T} . The w -line through g is the coarse equivalence class of the set $\{gw^m\}_{m \in \mathbb{Z}}$. This coarse equivalence class contains a unique geodesic l with endpoints on the boundary: $l^+ = gw^\infty$ and $l^- = gw^{-\infty}$. If w is cyclically reduced then g is actually a point on the line l .

Let $\mathcal{L}_{\underline{w}}$ be the line pattern generated by a multiword \underline{w} . The decomposition space $\mathbf{D}_{\underline{w}} = \mathbf{D}_{\mathcal{L}_{\underline{w}}}$ is the quotient of $\partial\mathcal{T}$ obtained by identifying the two points l^+ and l^- for each $l \in \mathcal{L}_{\underline{w}}$. Let $\delta_{\underline{w}}: \partial\mathcal{T} \rightarrow \mathbf{D}_{\underline{w}}$ be the quotient map. We will often drop the \underline{w} subscript when it is clear that we are working with a fixed multiword.

Note that a change of basis of F gives equivariant homeomorphisms of $\partial\mathcal{T}$ and of \mathbf{D} , so we can talk about these spaces without reference to a specific basis. It also makes sense to define $\partial F = \partial\mathcal{T}$.

The decomposition space is a perfect, metrizable, compact, Hausdorff topological space. When it is connected it is also locally connected and has topological dimension one.

By construction, distinct lines l and l' in \mathcal{L} never have a common endpoint in $\partial\mathcal{T}$. Thus, the preimage of a point in the decomposition space is either a single point in $\partial\mathcal{T}$ or the pair of endpoints of a line in the pattern.

Any homeomorphism of $\partial\mathcal{T}$ that preserves the collection of pairs of endpoints of lines of \mathcal{L} descends to a homeomorphism of the decomposition space. In particular, any quasi-isometry of \mathcal{T} that, up to coarse equivalence, preserves the line pattern will induce a homeomorphism of the decomposition space.

We will primarily be interested in the case that the decomposition space is connected. A *cut set* is a set whose complement is not connected. The cut set is *minimal* if no proper subset is a cut set. A *cut point* is a point that is a cut set. A two point cut set ought to be called a cut pair, but we will be interested in minimal cut pairs, so the term *cut pair* will be reserved for a two point cut set, neither point of which is a cut point.

If $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are cut pairs, we say $\{x_0, x_1\}$ *crosses* $\{y_0, y_1\}$ if x_0 and x_1 are in different complementary component of $\mathbf{D} \setminus \{y_0, y_1\}$. Since we assume that no point of a cut pair is a cut point, crossing is symmetric: $\{x_0, x_1\}$ crosses $\{y_0, y_1\}$ if and only if $\{y_0, y_1\}$ crosses $\{x_0, x_1\}$. Thus, we can say $\{x_0, x_1\}$ and $\{y_0, y_1\}$ *cross*.

Note that a cut point can not be crossed by a cut pair: the cut pair would not disconnect the space unless one of its points was actually a cut point, but the term “cut pair” is reserved for two point cut sets that are minimal. Similarly, a cut pair with more than two complementary connected components can not be crossed by a cut pair.

2.2. A Line Pattern Lifted to a Finite Index Subgroup. Let G be a finite index subgroup of F . Let $1 = f_1, \dots, f_k$ be right coset representatives, so that $F = \coprod_{i=1 \dots k} Gf_i$. The map $\bar{\iota}: F \rightarrow G$ that sends gf_i to $g \in G$ is a quasi-isometry. It is a coarse inverse to the inclusion $\iota: G \rightarrow F$.

Let w_j be an element of a multiword \underline{w} . For each i and j there exists a minimal positive integer $a = a(i, j)$ such that $f_i w_j^a \bar{f}_i \in G$. Therefore, for any $h = gf_i$, we have:

$$\{h w_j^m\}_{m \in \mathbb{Z}} \stackrel{c}{=} \{h w_j^{am}\}_{m \in \mathbb{Z}} = \{g f_i w_j^{am}\}_{m \in \mathbb{Z}} \stackrel{c}{=} \{g (f_i w_j^a \bar{f}_i)^m\}_{m \in \mathbb{Z}} \subset G$$

Thus, if $h \in Gf_i$, the map $\bar{\iota}$ sends the w_j -line through h in F to a set of points coarsely equivalent to the $(f_i w_j^a \bar{f}_i)$ -line through $\bar{\iota}(h)$ in G .

Consider the set of conjugacy classes in G given by $\{[f_i w_j^{a(i,j)} \bar{f}_i] \mid \text{for all } i, j\}$. Pick a cyclically reduced representative from each distinct conjugacy class. This gives a multiword \underline{w}' that generates a line pattern in G that is equivalent to the image of $\mathcal{L}_{\underline{w}}$ under $\bar{\iota}$. We call this the *line pattern $\mathcal{L}_{\underline{w}}$ lifted to G* , and call \underline{w}' the *multiword \underline{w} lifted to G* .

In topological terms, think of F as the fundamental group of a surface (with boundary). The multiword \underline{w} can be represented by a multicurve on the surface. Let G be the fundamental group of a covering surface. For each curve in the

multicurve, take all of its lifts to the G surface. The collection of all of these is a multicurve representing \underline{w}' .

The lifted line pattern is independent of the choice of coset representatives and choice of \underline{w}' , and $\bar{\iota}$ takes each line of $\mathcal{L}_{\underline{w}}$ to a set coarsely equivalent to a line in the lifted pattern, and similarly for ι . Therefore, $\bar{\iota}$ extends to a G -equivariant homeomorphism $\partial F \rightarrow \partial G$ that preserves pairs of endpoints of lines in the pattern. This homeomorphism descends to a G -equivariant homeomorphism of decomposition spaces.

2.3. Whitehead Graphs. The primary tool for understanding the topology of the decomposition space associated to a line pattern is the generalized Whitehead graph of a generating multiword. This machinery was developed in [7]. In this section we will recall the relevant definitions and results, but see [7] for details.

The Whitehead graph $\text{Wh}(\ast) = \text{Wh}_{\mathcal{B}}(\ast)\{w\}$ of a cyclically reduced word w with respect to a basis \mathcal{B} of F is a graph with $2n$ vertices labeled with the elements of \mathcal{B} and their inverses. One edge joins vertex x to vertex y for each occurrence of $\bar{x}y$ in the word w as a cyclic word.

Similarly, the Whitehead graph of a multiword is obtained by adding edges for each element of the multiword. The number of edges is therefore equal to the sum of the word lengths with respect to \mathcal{B} .

Whitehead's Algorithm [20] gives an algorithm for reducing the number of edges of the graph by applying a certain collection of Whitehead automorphisms. The complexity, the number of edges, can be reduced monotonically down to some minimum. A particularly important observation is that if some Whitehead graph $\text{Wh}_{\mathcal{B}}(\ast)$ is connected and has a cut vertex, a vertex whose removal disconnects the graph, then this Whitehead graph is not minimal complexity.

An easy consequence of Whitehead's results is the following:

Proposition 2.1. *Given a multiword \underline{w} in a free group F , the following are equivalent:*

- (1) *Some Whitehead graph for \underline{w} is not connected.*
- (2) *Every minimal Whitehead graph for \underline{w} is not connected.*
- (3) *F splits freely rel \underline{w} .*

The next three proclamations relate Proposition 2.1 to the decomposition space. They come from work of Otal [15, Prop 2.1], and can also be found in the thesis of Reiner Martin [13, Theorem 49].

Lemma 2.2. [7, Lemma 4.3] *If for some choice of basis $\text{Wh}(\ast)$ is disconnected, then \mathcal{D} is disconnected.*

Lemma 2.3 ([7, Lemma 4.4]). *Suppose there exists a free basis \mathcal{B} of F such that $\text{Wh}_{\mathcal{B}}(\ast)$ is connected without cut vertices. Let \mathcal{T} be the Cayley graph of F corresponding to \mathcal{B} . Pick any edge e in \mathcal{T} . Let \ast and v be the endpoints of e . Let \hat{A} be the collection of points $\xi \in \partial \mathcal{T}$ such that v is on $[\ast, \xi]$. The set $A = \delta(\hat{A})$ is connected in \mathcal{D} .*

Let $\{l_1, \dots, l_k\} \subset \mathcal{L}$ be the set of lines that contain e . For each i , the two endpoints l_i^+ and l_i^- are identified in the decomposition space. Thus:

$$\delta(\hat{A}) \cap \delta(\hat{A}^c) = \bigcup_{i=1..k} \delta(l_i^+)$$

So, if $\text{Wh}(\ast)$ is connected without cut vertices, then for any edge e in \mathcal{T} the boundaries at infinity of the two connected components of $\mathcal{T} \setminus e$ correspond to connected sets in the decomposition space. Since $\text{Wh}(\ast)$ is connected there is also at least one line in \mathcal{L} crossing e , so these connected sets have a point in common.

Corollary 2.4. *Suppose $\text{Wh}(\ast)$ has no cut vertices. The decomposition space is connected if and only if $\text{Wh}(\ast)$ is connected. Furthermore, if \mathbf{D} is connected it is also locally connected.*

Moreover, if none of the $\delta(l_i^+)$ is a cut point of \mathbf{D} then, in fact,

$$\delta(\hat{A}) \setminus \bigcup_{i=1..k} \delta(l_i^+)$$

is a connected subset of \mathbf{D} .

Assumption. Unless otherwise noted, throughout the paper we will assume that we are working with a multiword such that the decomposition space is connected, and that we have chosen a basis for the free group such that the corresponding Whitehead graph is connected without cut vertices. Equivalently, F does not split freely relative to the multiword.

There is no loss of generality from this assumption. If the decomposition space were not connected we would first pass to a maximal free splitting of F relative to the multiword and then deal with the free factors separately. Once the decomposition space is connected we can use Whitehead's algorithm to produce a basis such that the Whitehead graph is connected without cut vertices. For computational purpose it would probably be wise to choose a basis that gives a minimal complexity Whitehead graph, but all of our arguments still work with only the "connected without cut vertices" assumption.

The classical Whitehead graph generalizes: if \mathcal{L} is a line pattern in F and \mathcal{X} is a connected subset of $\overline{\mathcal{T}}$ we define the Whitehead graph $\text{Wh}_{\mathcal{G}}(\mathcal{X})\{\mathcal{L}\}$ to be the graph with vertices in bijection with connected components of $\overline{\mathcal{T}} \setminus \mathcal{X}$ and one edge joining v and v' for each line of \mathcal{L} that has one endpoint in the component corresponding to v and the other in the component corresponding to v' . We will omit \mathcal{B} and \mathcal{L} when they are clear and just write $\text{Wh}(\mathcal{X})$. The notation $\text{Wh}(\ast)$ for the classical Whitehead graph just means the Whitehead graph at a vertex \ast , and because the line pattern is equivariant we get the same graph for any vertex.

Let $\mathcal{X} \subset \mathcal{Y} \subset \overline{\mathcal{T}}$. Let e be an edge of \mathcal{T} incident to exactly one vertex of \mathcal{X} .

The edge e corresponds to a vertex in $\text{Wh}(\mathcal{X})$. The graph $\text{Wh}(\mathcal{X}) \setminus e$ is obtained from $\text{Wh}(\mathcal{X})$ by deleting this vertex, but retaining the incident edges as *loose ends at e* .

If v is a vertex of \mathcal{T} that is distance 1 from \mathcal{X} , then there is a unique edge e with one endpoint equal to v and the other in \mathcal{X} . Define $\text{Wh}(\mathcal{X}) \setminus v = \text{Wh}(\mathcal{X}) \setminus e$.

Similarly, $\text{Wh}(\mathcal{X}) \setminus \mathcal{Y}$ is obtained from $\text{Wh}(\mathcal{X})$ by deleting each vertex of $\text{Wh}(\mathcal{X})$ that corresponds to an edge in \mathcal{Y} . Visualizing Whitehead graphs in the tree, $\text{Wh}(\mathcal{X}) \setminus \mathcal{Y}$ is the portion of $\text{Wh}(\mathcal{Y})$ that passes through the set \mathcal{X} .

An example follows (Example 2.6), but see [7] for a more detailed account.

Lemma 2.5 ([7, Lemma 4.9]). *Let S be a nonempty, finite subset of \mathbf{D} whose preimage in $\partial\mathcal{T}$ is more than one point. Let \mathcal{H} be the convex hull of $\delta^{-1}(S)$. There is a bijection between connected components of $\text{Wh}(\mathcal{H})$ and connected components of $\mathbf{D} \setminus S$.*

Let S be a finite cut set whose preimage in $\partial\mathcal{T}$ is a pair of points, so that \mathcal{H} is a line. The fact that $\text{Wh}(\ast)$ is connected without cut vertices implies that every vertex of $\text{Wh}(\mathcal{H})$ belongs to a component that limits to both boundary points of \mathcal{H} . It follows that if $\{x, y\}$ is a cut pair in \mathbf{D} , then for any small connected neighborhood N of x in \mathbf{D} the number of components of $N \setminus x$ is equal to the number of components of $\mathbf{D} \setminus \{x, y\}$, which is the number of components of $\text{Wh}(\mathcal{H})$.

It is easy to recognize whether a given element $g \in F$ gives rise to a cut set $\delta(\{\bar{g}^\infty, g^\infty\}) \in \mathbf{D}$. Just consider the quotient of $\text{Wh}([\bar{g}^\infty, g^\infty])$ by the g -action. This can be accomplished by looking at the quotient of a finite graph. If $g = ha$ for $a \in \mathcal{B} \cup \bar{\mathcal{B}}$, let $[\ast, g)$ denote the segment $[\ast, h] \subset [\ast, g]$. Then $\text{Wh}([\ast, g)) \setminus [\bar{g}^\infty, g^\infty]$ is almost exactly the portion of $\text{Wh}([\bar{g}^\infty, g^\infty])$ over $[\ast, g)$. The only discrepancy is that if $g \in \underline{w}$ then the g -line through \ast does not contribute an edge to $\text{Wh}([\bar{g}^\infty, g^\infty])$, but there is a loose edge in $\text{Wh}([\ast, g)) \setminus [\bar{g}^\infty, g^\infty]$ that the g -action glues to itself to form a closed loop containing no vertices. We consider such a loop a trivial component of the quotient; throw it away, if it exists. The number of other components is the number of complementary components of $\delta(\{\bar{g}^\infty, g^\infty\})$ in \mathbf{D} .

Example 2.6. Consider the multiword $\underline{w} = \{ab, \bar{a}\bar{b}ab\}$ in $F = \langle a, b \rangle$. In all of these figures the a -direction is right, the \bar{a} direction is left, the b -direction is up, and the \bar{b} -direction is down.

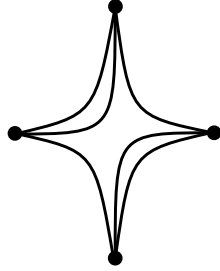
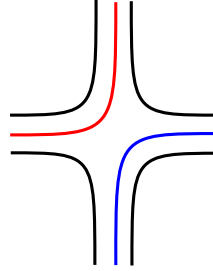
FIGURE 5. $\text{Wh}(\ast)$ FIGURE 6. $\text{Wh}(\ast)$ (loose)

Figure 5 shows the classical Whitehead graph $\text{Wh}(\ast)$. We think of this as a picture of the line pattern $\mathcal{L}_{\underline{w}}$ in a small neighborhood of the vertex $\ast \in \mathcal{T}$ representing the identity element of F . Note that it is connected without cut vertices.

Figure 6 shows the same Whitehead graph with "loose ends". The vertices have been opened up to indicate the group action. In this figure, the blue edge comes from the ab -line through \ast . The red edge comes from the ab -line through \bar{a} . The a -action takes the ab -line through \bar{a} to the ab -line through \ast . In the figure the a -action corresponds to a right shift, and the diagram is drawn so that a right shift would match up an end of the red edge to an end of the blue edge. Similarly, the b -action is a vertical shift, and should identify the ab -line through \ast , the blue edge, with the ab -line through b . Since \bar{a} and b differ by a multiple of ab , the ab -line through b is the same as the ab -line through \bar{a} , so, again, this is the red edge, and the diagram is drawn so that a vertical shift matches up an end of the blue edge to an end of the red edge. The four black edges come from the four distinct $\bar{a}\bar{b}ab$ -lines passing through \ast , and the edges are drawn in such a way that the vertical and horizontal shifts match up edges in the same way that the b -action and a -action, respectively, act on the collection of $\bar{a}\bar{b}ab$ -lines.

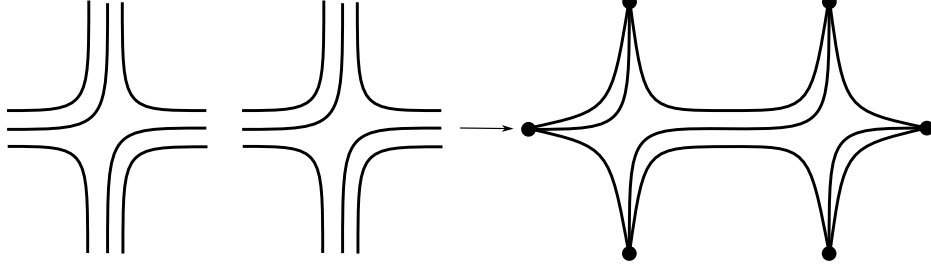


FIGURE 7. Splicing

Figure 7 illustrates how multiple copies of this loose ended Whitehead graph can be combined using Manning's splicing construction [12] to build Whitehead graphs over larger regions of \mathcal{T} . Here two copies of $\text{Wh}(\ast)$ splice together to make $\text{Wh}(\ast, a]$.

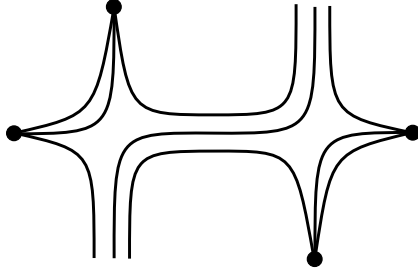
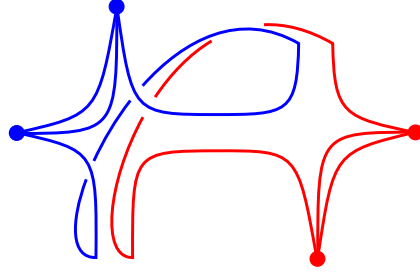
FIGURE 8.
 $\text{Wh}(\ast, ab) \setminus [(\overline{ab})^\infty, (ab)^\infty]$ FIGURE 9. nontrivial
components in the
quotient by ab -action

Figure 8 shows $\text{Wh}(\ast, ab) \setminus [(\overline{ab})^\infty, (ab)^\infty]$. This is the portion of the Whitehead graph of $\text{Wh}([(\overline{ab})^\infty, (ab)^\infty])$ that is visible on the interval $\ast, ab = \ast, a]$. (This is not quite right, as the edge with no vertices is actually the ab -line through \ast , which does not contribute an edge to $\text{Wh}([(\overline{ab})^\infty, (ab)^\infty])$.) This is a fundamental domain for the ab -action on $[(\overline{ab})^\infty, (ab)^\infty]$. The quotient of $\text{Wh}(\ast, a] \setminus [(\overline{ab})^\infty, (ab)^\infty]$ by the (ab) -action has two non-trivial components, as seen in Figure 9, so we conclude that $\delta(\{(\overline{ab})^\infty, (ab)^\infty\})$ is a cut set in \mathbf{D} . Furthermore, since $ab \in \underline{w}$, it is a cut point.

2.4. Cut Point and Cut Pair Detection and Approximation.

Lemma 2.7. *If $x \in \mathbf{D}$ is a cut point then the stabilizer of x is conjugate to $\langle g \rangle$ for an element $g \in \underline{w}$.*

Proof. Using an argument similar to Lemma 2.5, it is easy to see that $\delta^{-1}(x)$ must be two points. The result follows. \square

Lemma 2.8 (cf [7, Lemma 4.12]). *Let ξ_0 and ξ_1 be a pair of points in $\partial\mathcal{T}$ such that $\delta(\{\xi_0, \xi_1\})$ is a cut pair in \mathbf{D} and for each i we have $\xi_i = \delta^{-1}(\delta(\xi_i))$. There exist elements $g, h \in F$ and $a \in \mathcal{B} \cup \overline{\mathcal{B}}$ such that:*

- (1) the oriented edges $[h, ha]$ and $[hg, hga]$ belong to $[\xi_0, \xi_1] \cap [h\bar{g}^\infty, hg^\infty]$,
- (2) components of $\text{Wh}([ha, hg]) \setminus \{h, hga\}$ that are in different components of $\text{Wh}([\xi_0, \xi_1])$ are in different components of $\text{Wh}([h\bar{g}^\infty, hg^\infty])$, and
- (3) for each line $l \in \mathcal{L}$ crossing the edge $[h, ha]$, the lines l and $hg\bar{h}l$ (which crosses $[hg, hga]$) belong to the same component of $\text{Wh}([\xi_0, \xi_1])$.

Moreover, the conjugacy class of g contains an element whose length is uniformly bounded in terms of the rank of F , \underline{w} , and \mathcal{B} .

In other words, the cut pair $\delta(\{\xi_0, \xi_1\})$ can be approximated by a rational cut set $\delta(\{h\bar{g}^\infty, hg^\infty\})$. Item (1) says that $[\xi_0, \xi_1] \cap [h\bar{g}^\infty, hg^\infty]$ contains a copy of the fundamental domain for the $hg\bar{h}$ action on $[h\bar{g}^\infty, hg^\infty]$. Items (2) and (3) say that the $hg\bar{h}$ action behaves nicely with respect to connected components of $\mathbf{D} \setminus \delta(\{\xi_0, \xi_1\})$, at least if we look at the Whitehead graph on the overlap.

Proof. Assume $[\xi_0, \xi_1]$ is oriented from ξ_0 to ξ_1 .

Let n be the rank of the F . Let x be the maximum valence of $\text{Wh}(\ast)$ (which depends on the choice of \mathcal{B}). Let y be the x -th Bell number, the number of distinct partitions of x items into nonempty subsets. Let $z = 1 + (2n)^{y+2}$. Along any segment \mathcal{X} of $[\xi_0, \xi_1]$ of length z there is some $a \in \mathcal{B} \cup \bar{\mathcal{B}}$ such that there are at least $y + 2$ many directed a -edges in the segment. Fix the first of these, $e = [g_0, g_0a]$.

There are finitely many lines of \mathcal{L} that cross e . Fix a numbering of them $1, \dots, k$. Of course, $k \leq x$. Partition them into subsets according to the component of $\text{Wh}([\xi_0, \xi_1])$ to which they belong.

Consider an element $g' \in F$ such that the oriented edge $g'e$ is in \mathcal{X} . There is a bijection $l \mapsto g'l$ between lines of \mathcal{L} crossing e and lines of \mathcal{L} crossing $g'e$, so the numbering of the lines crossing e can be pushed forward to a numbering of the lines crossing $g'e$. We can also partition the lines crossing $g'e$ according to the component of $\text{Wh}([\xi_0, \xi_1])$ to which they belong.

There are at least $y + 1$ such g' , but at most y distinct partitions, so for some of these g' the partitions are the same. Thus, there exist g_1 and g_2 such that the oriented edges g_1e and g_2e are edges of $[\xi_0, \xi_1]$ (with g_2e between g_1e and ξ_1) and for each line $l \in \mathcal{L}$ crossing g_1e , the corresponding line $g_2\bar{g}_1l$ crossing g_2e is in the same component of $\text{Wh}([\xi_0, \xi_1])$.

The desired elements are $h = g_1g_0$ and $g = \bar{g}_0\bar{g}_1g_2g_0$.

Moreover, g is conjugate to an element \bar{g}_1g_2 of length at most z .

□

We can test an element g to see if it gives a cut point or cut pair in the decomposition space by counting non-trivial components of $\text{Wh}(\ast, g) \setminus [\bar{g}^\infty, g^\infty]$. If we test all elements of length less than the bound from Lemma 2.8 and find no cut points or cut pairs, then there are no cut points or cut pairs, so \underline{w} is rigid. Thus:

Corollary 2.9. *There is an algorithm to decide if (F, \underline{w}) is rigid.*

The algorithm described is far from efficient, and we will not spend any time in this paper verifying that a multiword is rigid. The interested reader is referred to [7] for a more detailed account. However, there is one prototypical family of examples of rigid multiwords worth mentioning:

Proposition 2.10 ([7, Section 6.2]). *If \underline{w} and \mathcal{B} are a multiword and basis of $F = F_n$ such that $\text{Wh}_{\mathcal{B}}(\ast)\{\underline{w}\}$ is the complete graph on $2n$ vertices, the graph with exactly one edge joining each pair of distinct vertices, then (F, \underline{w}) is rigid.*

Conversely, we can decide if (F, \underline{w}) is a QH-surface:

Lemma 2.11 ([15, Theorem 2], [7, Theorem 6.1]). *Given a \underline{w} multiword in a free group F , the following are equivalent:*

- (1) (F, \underline{w}) is a QH-surface.
- (2) \underline{w} consists of the boundary words of a compact surface with boundary with fundamental group F .
- (3) Some Whitehead graph for \underline{w} is a circle.
- (4) Every Whitehead graph for \underline{w} with no cut vertex is a circle.
- (5) Every minimal Whitehead graph for \underline{w} is a circle.
- (6) \mathbf{D} is a circle.
- (7) Every minimal cut set of \mathbf{D} has size two.

2.5. Normalization. Suppose Γ is a graph of groups decomposition of F with cyclic edge groups. It will be useful to normalize Γ without changing the non-cyclic vertex groups.

First subdivide each edge incident to a non-cyclic vertex so that the edge map into each non-cyclic vertex maps onto a maximal cyclic subgroup in the vertex. Now fold all the edges together that map into a common conjugacy class of maximal cyclic subgroup in a given non-cyclic vertex (changing stable letters, if necessary). Finally, since F is free, every edge groups maps onto a maximal cyclic subgroup in one of its vertex groups. Thus, any interior edges, edges adjacent to two cyclic vertices, can be collapsed.

We are left with a new graph of groups decomposition of F that is bipartite: vertex groups are either maximal cyclic subgroups or are non-cyclic. Cyclic vertex groups are adjacent only to non-cyclic vertex groups, and vice versa. Furthermore, each non-cyclic vertex has at most one edge mapping into any conjugacy class of maximal cyclic subgroup, and each edge map maps an edge group onto a maximal cyclic subgroup containing its image in the vertex group.

Figure 10 shows an example of this process. In the diagram all edges have cyclic stabilizers. Unlabeled vertices have cyclic stabilizers. Assume a generator has been fixed for each of the cyclic vertex and edge groups. Edge labels indicate the image of the generator of the edge stabilizer into the incident vertex group. Labels of 1 are omitted.

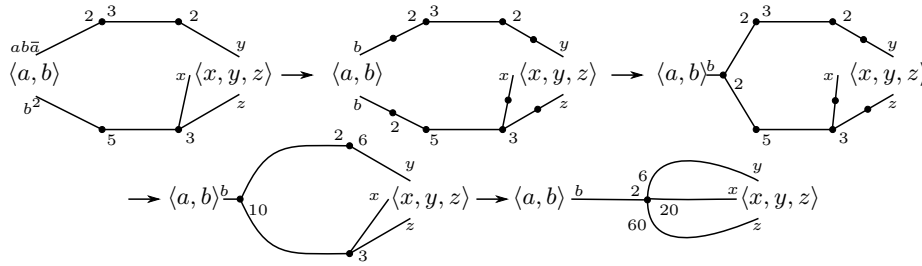


FIGURE 10. Normalization

3. SPLITTINGS

Otal [15] makes the following observation:

Lemma 3.1. *If F splits over $\langle g \rangle$ relative to \underline{w} then $\delta(\{\bar{g}^\infty, g^\infty\})$ is a cut point or cut pair in \mathbf{D} .*

Since rigid multiwords have no cut points or cut pairs in their decomposition spaces, this immediately gives:

Corollary 3.2. *If (F, \underline{w}) is rigid then there are no splittings of F relative to \underline{w} .*

To produce splittings we would like to know when the existence of cut points or cut pairs implies the existence of splittings.

3.1. Crossing Pairs and the Circle. In this subsection we give criteria for the decomposition space to be a circle.

In a circle, the only minimal cut sets are cut pairs, and every cut pair is crossed by a cut pair.

Lemma 3.3. *Suppose $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are crossing cut pairs in \mathbf{D} . Let A_0 and A_1 be the two connected components of $\mathbf{D} \setminus \{x_0, x_1\}$, and assume $y_0 \in A_0$ and $y_1 \in A_1$. Similarly, let B_0 and B_1 be the connected components of $\mathbf{D} \setminus \{y_0, y_1\}$, and assume $x_0 \in A_0$ and $x_1 \in A_1$. Then $\{x_0, y_0\}$ is a cut pair with connected components $C_0 = A_0 \cap B_0$ and $C_1 = A_1 \cup B_1 \cup \{x_1\} \cup \{y_1\}$.*

Proof. Since $\{x_0, x_1\}$ and $\{y_0, y_1\}$ cross, y_0 is a cut point of the connected set A_0 . Any neighborhood of y_0 contains a connected open set N such that $N \setminus y_0$ has exactly two connected components. Therefore, $A_0 \setminus y_0$ has exactly two connected components, which are $A_0 \cap B_0$ and $A_0 \cap B_1$. Similarly, the sets $A_1 \cap B_0$ and $A_1 \cap B_1$ are connected.

It follows that C_1 is a connected set and $\overline{C_1} \setminus C_1 = \{x_0, y_0\}$.

Now:

$$\{x_0, y_0\} \subset \overline{C_0} \setminus C_0 \subset \{x_0, x_1, y_0, y_1\}$$

We are done if x_1 and y_1 are not limit points of C_0 . Suppose, without loss of generality, that x_1 is a limit point of $C_0 = A_0 \cap B_0$. Since x_1 is also a limit point of $A_0 \cap B_1$ and $A_1 \cap B_1$, this implies that for any small neighborhood N of x_1 , the complement $N \setminus x_1$ has at least three connected components, which is a contradiction. \square

The following proposition is a generalization of a construction of Bowditch for boundaries of hyperbolic groups [4].

Proposition 3.4. *The decomposition space \mathbf{D} is a circle if and only if all of the following conditions are satisfied:*

- (1) \mathbf{D} is connected.
- (2) \mathbf{D} has no cut points.
- (3) \mathbf{D} has cut pairs.
- (4) Every cut pair in \mathbf{D} is crossed by a cut pair.

Proof. A circle satisfies these conditions, so one direction is clear.

Define a relation on \mathbf{D} by $x \sim y$ if $x = y$ or if $\{x, y\}$ is a cut pair.

Claim 3.4.1. \sim is an equivalence relation.

Proof of Claim. We must show transitivity.

Suppose x, y and z are distinct points with $x \sim y$ and $y \sim z$, so that $\{x, y\}$ and $\{y, z\}$ are cut pairs. By hypothesis, every cut pair is crossed by a cut pair.

Therefore, every cut pair has exactly two complementary connected components. Let A_0 and A_1 be the connected components of $\mathbf{D} \setminus \{x, y\}$. Let B_0 and B_1 be the connected components of $\mathbf{D} \setminus \{y, z\}$. Assume that $z \in A_0$ and $x \in B_0$. Arguing as in the proof of Lemma 3.3, $\mathbf{D} \setminus \{x, z\}$ has two connected components, $A_0 \cap B_0$ and $A_1 \cup B_1 \cup \{y\}$. \diamond

Claim 3.4.2. Equivalence classes are closed.

Proof of Claim. If an equivalence class consists of a single point it is closed, so suppose $[x]$ is not a single point.

Let $(y_i) \rightarrow y$ for $y_i \in [x]$. Choose points $\xi \in \delta^{-1}(y)$ and $\xi_i \in \delta^{-1}(y_i)$. After passing to a subsequence, $(\xi_i) \rightarrow \xi$ in $\partial\mathcal{T}$.

Pick $\eta \in \delta^{-1}(x)$. For each i , $\text{Wh}([\eta, \xi_i])$ has two components.

Now consider $[\eta, \xi]$. Number the vertices v_j along this geodesic with consecutive integers, increasing in the ξ direction, where v_0 may be any vertex on $[\eta, \xi]$. Since $(\xi_i) \rightarrow \xi$, for every j there is an I_j such that for all $i > I_j$, we have $[\eta, v_{j+1}] \subset [\eta, \xi_i]$. Therefore, $\text{Wh}([\eta, v_j]) \setminus [\eta, \xi] = \text{Wh}([\eta, v_j]) \setminus [\eta, \xi_i]$.

Thus, for all j , the Whitehead graph $\text{Wh}([\eta, v_j]) \setminus [\eta, \xi]$ has two components, which implies $\delta(\{\eta, \xi\}) = \{x, y\}$ is a cut pair. Hence, $y \in [x]$, and $[x]$ is closed. \diamond

Claim 3.4.3. All of \mathbf{D} is in one equivalence class.

Proof of Claim. We have assumed that a cut pair exists, so there is an equivalence class $[x]$ consisting of more than one point. Suppose that $[x]$ is not all of \mathbf{D} .

Let U be a connected component of $\mathbf{D} \setminus [x]$. Since \mathbf{D} is locally connected and $[x]$ is closed, U is open in \mathbf{D} . Since \mathbf{D} is connected without cut points, U has at least two limit points in $[x]$. Pick distinct points y_0 and y_1 in $\overline{U} \cap [x]$. These points are a cut pair, and $\mathbf{D} \setminus \{y_0, y_1\}$ has exactly two connected components, A_0 and A_1 . Assume $U \subset A_0$.

Let $\{z_0, z_1\}$ be a cut pair crossing $\{y_0, y_1\}$ with complementary connected components B_0 and B_1 . Assume $z_0 \in A_0$, $z_1 \in A_1$, $y_0 \in B_0$ and $y_1 \in B_1$.

By Lemma 3.3, y_0 and y_1 are in $[x] \subset \mathbf{D} \setminus U$. Thus, U is contained in B_ϵ , where ϵ is either 0 or 1. Since $U \subset A_0$, we have $U \subset A_0 \cap B_\epsilon$.

Now, $\{y_\epsilon, z_0\}$ is a cut pair whose connected components are $C_0 = A_0 \cap B_\epsilon$ and $C_1 = A_1 \cup B_{1+\epsilon} \cup \{y_{1+\epsilon}\} \cup \{z_1\}$ (subscripts modulo 2). However, U , and hence C_0 , has $y_{1+\epsilon} \in C_1$ as a limit point, which is a contradiction. Thus, $[x] = \mathbf{D}$. \diamond

Since $[x] = \mathbf{D}$, every point of \mathbf{D} is a member of a cut pair, and the only minimal cut sets are of size two. It then follows from Lemma 2.11 that \mathbf{D} is a circle. \square

Corollary 3.5. *If \mathbf{D} is connected, not rigid, and not a circle, then \mathbf{D} contains cut points or uncrossed cut pairs.*

3.2. Uncrossed Cut Pairs. A cut point in \mathbf{D} is of the form $\delta(\{\bar{w}^\infty, w^\infty\})$ for some $w \in \underline{w}$. Therefore, the preimage $\delta^{-1}(\delta(\{\bar{w}^\infty, w^\infty\}))$ is the pair of points $\{\bar{w}^\infty, w^\infty\} \subset \partial\mathcal{T}$. The convex hull of these two points is a line on which the element w acts cocompactly. We will show that uncrossed cut pairs have a similar form.

In general the preimage in $\partial\mathcal{T}$ of a cut pair in \mathbf{D} could have as many as four points, so the convex hull may not be a line. Even if a cut pair has a preimage

consisting of two points, it may not have a group element acting cocompactly on the convex hull of these two points.

An easy argument, Lemma 3.6, shows that the convex hull of the preimage of an uncrossed cut pair is a line.

Recall that Lemma 2.8 says that for any cut pair such that the convex hull of the preimage is a line, we can find a group element g of uniformly bounded length whose axis approximates the hull in a certain strong sense. Proposition 3.7 shows that if the cut pair is uncrossed then the axis of g and the convex hull of the preimage coincide.

Lemma 3.6. *The preimage of an uncrossed cut pair is two points, so its convex hull is a line.*

Proof. Suppose we have a cut pair $\{x_0, x_1\} \subset \mathbf{D}$ that is not crossed by any other cut pair. Suppose $\delta^{-1}(x_0) = \{l^-, l^+\}$ for some w -line $l \in \mathcal{L}$ through some vertex h in \mathcal{T} . Acting on everything by \bar{h} and renaming, we may assume that l is the w -line through $*$.

By an argument similar to the proof of Lemma 3.3, for some i and j , the pair $\{w^{-i}x_1, w^jx_1\}$ is a cut pair crossing $\{x_0, x_1\}$, contrary to hypothesis. \square

Proposition 3.7. *If $\{x_0, x_1\}$ is an uncrossed cut pair and g and h are the elements from Lemma 2.8 such that the axis of $hg\bar{h}$ approximates the convex hull of $\delta^{-1}(\{x_0, x_1\})$, then $\delta(h\bar{g}^\infty) = x_0$ and $\delta(hg^\infty) = x_1$.*

Proof. Assume h is trivial by replacing $\{x_0, x_1\}$ with $\{\bar{h}x_0, \bar{h}x_1\}$ and renaming.

Let $\xi_i = \delta^{-1}(x_i)$, which is a single point by Lemma 3.6.

Suppose $g^\infty \neq \xi_1$. Consider the last vertex $v \in \mathcal{T}$ in the positive g direction such that $[\bar{g}^\infty, g^\infty]$ and $[\xi_0, \xi_1]$ overlap. Let e_o be the edge incident to v contained in the overlap. Let e_g be the edge leading to g^∞ . Let e_ξ be the edge leading to ξ_1 .

The Whitehead graph $\text{Wh}(v) \setminus \{e_o, e_g, e_\xi\}$ has four groups of components: Group A consisting of components that go out e_o and e_ξ but not e_g . Group B consisting of components that go out e_o and e_g but not e_ξ . Group C consisting of components that go out e_g and e_ξ but not e_o . Group D consisting of components that go out all three.

We further divide A into groups A' and A'' . Group A'' consists of those components that are contained in the component of $\text{Wh}([\xi_0, \xi_1])$ containing g^∞ . Group A' consists of the others.

All of the components in groups A'' , B , C , and D belong to a single component of $\text{Wh}([\xi_0, \xi_1])$, the one containing g^∞ . Since $\text{Wh}([\xi_0, \xi_1])$ has at least two components, A' is non-empty.

On the other hand, all of the components in groups A , C , and D belong to a single component of $\text{Wh}([\bar{g}^\infty, g^\infty])$, the one containing ξ_1 . Now, components of $\text{Wh}(\text{overlap})$ that are in separate $\text{Wh}([\xi_0, \xi_1])$ components must remain in separate $\text{Wh}([\bar{g}^\infty, g^\infty])$ components. Thus, anything in the A or D groups must belong to a common $\text{Wh}([\xi_0, \xi_1])$ component. However, the A' group pieces do not belong to the g^∞ component of $\text{Wh}([\xi_0, \xi_1])$, while the A'' and D groups do. Since the A' group is non-empty, the A'' and D groups must be empty. Also, the components in the A' group all belong to a common $\text{Wh}([\xi_0, \xi_1])$ component. Thus, the cut set $\{x_0, x_1\}$ has exactly two complementary components, the one containing g^∞ , and the one not containing g^∞ .

The C group can not be empty or the e_o edge would give a cut vertex in the Whitehead graph at v . Similarly, the B group can not be empty or the e_ξ edge would give a cut vertex.

Everything in the C group, as it continues towards ξ_1 , must remain separate from things in the A' group, because the C group belongs in the g^∞ component of $\text{Wh}([\xi_0, \xi_1])$, while the A' group does not. Similarly, pieces in the C group remain separate from pieces in the B group as they continue towards g^∞ , since the C group belongs to the component of $\text{Wh}([\bar{g}^\infty, g^\infty])$ containing ξ_1 , while things in the B group do not. This shows that the Whitehead graph $\text{Wh}([\xi_1, g^\infty])$ has at least two components, because the pieces from C and the pieces from $A' \cup B$ must be in separate components. Thus, $\delta(\{\xi_1, g^\infty\})$ is a cut pair.

There exists some number k such that $\bar{g}^k v \in \text{overlap}$, and $\delta(\{\bar{g}^k \xi_1, g^\infty\})$ is also a cut pair. Since the g action on the overlap preserves components of $\text{Wh}([\xi_0, \xi_1])$, the edge $\bar{g}^k e_\xi$ corresponds to a vertex in the component of $\text{Wh}([\xi_0, \xi_1])$ not containing g^∞ .

This gives us a contradiction: $\delta(\{\bar{g}^k \xi_1, g^\infty\})$ is a cut pair with one point in the g^∞ component of $\mathbf{D} \setminus \{x_0, x_1\}$ and one point in another, so $\{x_0, x_1\}$ is crossed.

Thus, $\xi_1 = g^\infty$. A similar argument shows $\xi_0 = \bar{g}^\infty$. □

Proposition 3.8. *There are only finitely many orbits of cut points and uncrossed cut pairs.*

Proof. For cut points this is clear, since by Lemma 2.7 the stabilizer of a cut point is generated by a conjugate of a word in w .

By Proposition 3.7 there is a member of each orbit of uncrossed cut pairs that is stabilized by an element of bounded length provided by Lemma 2.8. □

Proposition 3.9. *If \underline{w} is a multiword in F such that the decomposition space is connected and not rigid, then either:*

- (1) *the decomposition space is a circle, or*
- (2) *there is an indivisible element $g \in F$ such that $\delta(\{\bar{g}^\infty, g^\infty\})$ is a cut set that is not crossed by any cut pair.*

Proof. If we are not in case (1) then by Corollary 3.5 there exist cut points or uncrossed cut pairs. Take g to be the generator of the stabilizer of such a cut set. For a cut point one may assume g is in the multiword. For an uncrossed cut pair the existence of such a g is furnished by Lemma 3.6. □

3.3. The Splitting Criterion.

Proposition 3.10. *Consider a non-empty equivariant collection of cut sets $\{S_i\}_{i \in I}$ in \mathbf{D} satisfying the following conditions:*

- (1) *For each $i \in I$ there is a $g_i \in F$ such that $S_i = \delta(\{\bar{g}_i^\infty, g_i^\infty\})$.*
- (2) *The cut sets are pairwise non-crossing.*
- (3) *The set $\{S_i\}_{i \in I}$ is a union of finitely many F orbits.*

Then F splits as a graph of groups $\text{rel } \underline{w}$ with cyclic edge stabilizers. The vertex set is bipartite, with Type 1 vertices stabilized by maximal cyclic subgroups containing the g_i , and Type 2 vertices stabilized by non-cyclic subgroups.

Proposition 3.10 is an easy generalization of a construction of Otal [15], presented as Proposition 3.11, who proves it in the case that $\{S_i\}_{i \in I}$ is a single orbit of cut

points. At one point we use a line pattern argument to replace Otal's Lemma 3.3. We add some remarks to indicate the points where the proof generalizes. We also use the construction in this proof to make some additional conclusions, which follow as separate proclamations.

Proposition 3.11. *Suppose g is an indivisible element such that $S = \delta(\{\bar{g}^\infty, g^\infty\})$ is a cut point. Then F splits over $\langle g \rangle$ relative to \underline{w} .*

Proof. Consider translates hS of S by the group action. None of these cross each other, so the orbit of S has a partial ordering relative to S defined by $h'S < hS$ if $h'S$ separates S from hS , that is, if S and hS are in different components of $\mathbf{D} \setminus h'S$.

Remark. Note that such a partial ordering exists for any collection of disjoint cut sets that pairwise do not cross each other, even if they are not cut points or if they belong to more than one orbit.

For any translate hS , there are only finitely many other translates of S that separate hS from S . Therefore, there are *minimal* translates hS that are not separated from S by any translate of S .

Remark. Otal has a lemma to prove minimal translates exist. We can prove it with line patterns: Suppose that each cut set S_i is stabilized by a cyclic group $\langle g_i \rangle$. For S_2 to separate S_1 from S_3 , it is necessary that the axis of g_2 in \mathcal{T} intersects the finite geodesic segment joining the axis of g_1 to the axis of g_3 . The number of axes of conjugates of g_2 through any vertex is the length of the cyclic reduction of g_2 . If there are only finitely orbits of cut sets, then only finitely many axes intersect any compact set in \mathcal{T} , so only finitely many S_2 can separate S_1 from S_3 .

Define a simplicial tree on which F acts without inversions as follows. The tree has two types of vertices. The Type 1 vertices are bijection with the orbit of S . Given a Type 1 vertex, there is a corresponding cut set hS . This has finitely many complementary connected components. For each of these components, consider the subset of the orbit consisting of hS and all $h'S$ in the component such that $h'S$ is minimal with respect to hS . This subset forms a Type 2 vertex adjacent to the vertex hS .

The quotient of this tree by the F -action contains a single vertex of Type 1 for each orbit of cut set (one, in this case), and some finite number of adjacent Type 2 vertices, and the stabilizer of the Type 1 vertex is $\langle g \rangle$. Thus, we have finite graph of groups decomposition of F , and all edge stabilizers are subgroups of $\langle g \rangle$.

The generators of the line pattern must be conjugate into the vertex groups, otherwise we would have a line in the pattern crossing from one component of $\text{Wh}([\bar{g}^\infty, g^\infty])$ to another, which is absurd. \square

Combining Proposition 3.10 and Proposition 3.9 gives us a splitting theorem:

Theorem 3.12 (Splitting Theorem). *If \underline{w} is a multiword in F such that the corresponding decomposition space is connected and not rigid then either (F, \underline{w}) is a sphere with three boundary components or there exists a splitting of F over \mathbb{Z} relative to \underline{w} .*

Proof. The condition in case (2) of Proposition 3.9 satisfies the hypothesis of Proposition 3.10, so we get a splitting in this case.

In case (1) of Proposition 3.9 the decomposition space is a circle. This means that (F, \underline{w}) is a surface with its boundary components. Either this is a sphere with three boundary components or there exists an essential simple closed curve in the surface that is not boundary parallel, which gives a relative splitting. \square

3.4. Refining Splittings. Theorem 3.12 tells us when we can split F rel \underline{w} . In this subsection we determine when a splitting can be refined.

Lemma 3.13. *Let Γ be a graph of groups decomposition of F rel \underline{w} produced by Proposition 3.10. Let $\mathcal{D}(\Gamma)$ be the bipartite tree constructed in the proof, which is the Bass-Serre tree of the graph of groups decomposition. There is a natural F -equivariant surjective quasi-map $\pi_\Gamma: \mathbf{D} \rightarrow \overline{\mathcal{D}(\Gamma)}$.*

Proof. If $x \in S_i$ then define $\pi_\Gamma(x)$ to be the corresponding Type 1 vertex.

Otherwise, define a partial order relative to x on the set $\{S_i\}_{i \in I}$ by $S_i < S_j$ if x and S_j are in different components of S_i . There may or may not be minimal elements with respect to this partial order. First suppose there is a minimal element S_i . The point x is in one of the complementary components of S_i . Associated to this complementary component is a Type 2 vertex consisting of the other cut sets in the collection that are in the component and S_i -minimal. These cut sets are also x -minimal, since if S_j separates x from S_k it would also separate S_i from S_k . Define $\pi_\Gamma(x)$ to be the closed star of this Type 2 vertex.

Note that if $x = \delta(\{\bar{w}^\infty, w^\infty\})$ then, as in the proof of Proposition 3.11, there do exist minimal elements, because if S_i separates x from S_j it is necessary that the axis of g_i intersects the finite geodesic segment joining the axis of g_j to the convex hull of $\{\bar{w}^\infty, w^\infty\}$. By construction, there are only finitely many such axes.

Now suppose there are no minimal elements for the x -order. Let $\xi = \delta^{-1}(x) \in \partial\mathcal{T}$. There is a sequence $S_{i_1} > S_{i_2} > \dots$. Define $\pi_\Gamma(x)$ to be the boundary point in $\partial\mathcal{D}(\Gamma)$ to which the π_Γ images of the S_{i_j} converge. We show this is well defined. Passing to a subsequence, we may assume that the axes of the corresponding g_{i_j} are disjoint in \mathcal{T} . For each S_{i_j} there is an edge $e_{i_j} \in \mathcal{T}$ that is adjacent to the axis of g_{i_j} and separates the axis from ξ . Furthermore, for $k > j$ the axis of g_{i_k} is on the ξ -side of e_{i_j} . Thus, the edges e_{i_j} are distinct edges on a geodesic ray in \mathcal{T} converging to ξ . Now $\pi_\Gamma(x)$ is well defined since any other geodesic ray converging to ξ eventually coincides with this one, so that the tails of any two sequences $S_{i_1} > S_{i_2} > \dots$ and $S'_{i_1} > S'_{i_2} > \dots$ can be merged into a single ordered sequence.

It is clear that the map is equivariant and onto $\mathcal{D}(\Gamma)$. Let η be a point in $\partial\mathcal{D}(\Gamma)$. Pick a sequence of Type 1 vertices converging to η , and let S_{i_1}, S_{i_2}, \dots be the corresponding cut sets. By construction, for any $j < k < l$ we have that S_{i_k} separates S_{i_j} from S_{i_l} . As in the previous paragraph, from such a sequence we can pick a sequence of edges that lie on a geodesic ray in \mathcal{T} , converging to a point $\xi \in \partial\mathcal{T}$. Then $\pi_\Gamma(\delta(\xi)) = \eta$. \square

Definition 3.14. Let Γ be a graph of groups decompositions of F rel \underline{w} with cyclic edge stabilizers. Define the *augmented multiword*, denoted $\text{Aug}_\Gamma(\underline{w})$, to be the multiword in F consisting of words of \underline{w} together with generators of each of the edge groups of Γ .

Recall that we consider multiwords equivalent if they determine the same set of conjugacy classes of maximal cyclic subgroups. The number of words in $\text{Aug}_\Gamma(\underline{w})$ is not necessarily the number of words of \underline{w} plus the number of edges of Γ ; we only

need to add a new word if an edge stabilizer is not contained in a conjugate of $\langle w \rangle$ for any $w \in \underline{w}$.

Adding new words to a multiword results in a smaller decomposition space.

Definition 3.15. Define the *augmentation map*, α_Γ , to be the quotient map:

$$\alpha_\Gamma: \mathbf{D}_{\underline{w}} \rightarrow \mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$$

Note that $\delta_{\text{Aug}_\Gamma(\underline{w})} = \alpha_\Gamma \circ \delta_{\underline{w}}$.

Definition 3.16. Let G be a vertex stabilizer of a graph of groups decomposition Γ of F rel \underline{w} with cyclic edge stabilizers. If G is non-cyclic, define the *induced multiword* in G , denoted $\text{Ind}_\Gamma^G(\underline{w})$, to be the multiword in G consisting of any conjugates of elements of \underline{w} , along with generators of the maximal cyclic subgroups into which the incident edge groups include.

The point here is that a splitting of G rel the induced multiword should give a refinement of the graph of groups decomposition. This is why we add the incident edge groups, to ensure that a splitting of G will be compatible with the splitting we already have. The following lemmas make this precise.

Lemma 3.17. *Let G be a non-cyclic vertex group of Γ . The decomposition space $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ of G corresponding to $\text{Ind}_\Gamma^G(\underline{w})$ embeds naturally into the decomposition space $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$ of F corresponding to $\text{Aug}_\Gamma(\underline{w})$.*

Proof. The inclusion $\iota: G \hookrightarrow F$ is a quasi-isometric embedding, so it extends to a topological embedding $\partial\iota: \partial G \hookrightarrow \partial F$. The equivalence relation on ∂G coming from $\text{Ind}_\Gamma^G(\underline{w})$ is the restriction to $\partial\iota(\partial G)$ of the equivalence relation on ∂F coming from $\text{Aug}_\Gamma(\underline{w})$. \square

Theorem 3.18 (Structure Theorem for Decomposition Spaces). *The decomposition space $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$ is a compactified tree of decomposition spaces of the vertex groups of Γ .*

Proof. According to Lemma 3.13 there is an F -equivariant surjective quasi-map $\pi_\Gamma: \mathbf{D}_{\text{Aug}_\Gamma(\underline{w})} \rightarrow \mathcal{D}(\Gamma)$. The preimage of a Type 1 vertex of $\mathcal{D}(\Gamma)$ is a cut point in $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$. The preimage of the closed star of a Type 2 vertex stabilized by G is the copy of $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ in $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$. \square

Lemma 3.19. *Let Γ be a graph of groups decomposition of F rel \underline{w} with cyclic edge groups. Let G be a non-cyclic vertex of Γ . The decomposition space $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ of G with respect to $\text{Ind}_\Gamma^G(\underline{w})$ is connected.*

Proof. By Corollary 2.4, if $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ is not connected there is a free splitting of G rel $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$. This gives a free splitting of F rel $\text{Aug}_\Gamma(\underline{w})$, which implies that $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$ is not connected. This is not possible since $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$ is a quotient of $\mathbf{D}_{\underline{w}}$, which, by our standing assumption, is connected. \square

Lemma 3.20. *Let Γ be a graph of groups decomposition of F rel \underline{w} with cyclic edge groups. Let G be a non-cyclic vertex of Γ such that $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ is rigid. Let $\langle g \rangle$ be the stabilizer of an edge incident to G . Then $S = \delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ is either a cut point or uncrossed cut pair of $\mathbf{D}_{\underline{w}}$.*

Proof. S is a cut point if a root of a conjugate of g is in \underline{w} and is a cut pair otherwise.

If $\mathbf{D}_{\underline{w}} \setminus S$ has at least three components then S is uncrossed, so assume there are exactly two components.

If S is crossed then there is another cut pair R of $\mathbf{D}_{\underline{w}}$ whose points are in different components of $\mathbf{D}_{\underline{w}} \setminus S$. Conversely, R is crossed by S , so the cut pair R also has exactly two complementary components, A_0 and A_1 .

Let $\{S_i\}_{i \in I}$ be the collection of cut points and cut pairs corresponding to the edges of Γ . Let $J = \{j \mid S_j \text{ crosses or intersects } R\}$. Then $\alpha_\Gamma(R \cup \{S_j\}_{j \in J}) \cap \mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ is a cut pair in $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$, contradicting rigidity. \square

We will use the following kind of collections several times, so we give them a name.

Definition 3.21. An *uncrossed collection* in $\mathbf{D}_{\underline{w}}$ is a non-empty union of orbits of cut points and uncrossed cut pairs.

Note that an uncrossed collection satisfies the hypotheses of Proposition 3.10, since by Proposition 3.8 there are only finitely many orbits of cut points or uncrossed cut pairs. Thus, for any uncrossed collection there is a corresponding graph of groups decomposition of $F \text{ rel } \underline{w}$.

Lemma 3.22 (Universality of Uncrossed Splittings). *Let $\{S_i\}_{i \in I}$ be an uncrossed collection. Let Γ be the corresponding graph of groups decomposition of F . The stabilizer of a cut point or cut pair of $\mathbf{D}_{\underline{w}}$ is elliptic in Γ .*

Proof. Let S be a cut point or cut pair. By considering the F -action on the convex hull of $\delta^{-1}(S)$ it is clear that the stabilizer of S is either trivial or a maximal cyclic subgroup. If its stabilizer is trivial we are done, so assume its stabilizer is $\langle g \rangle$.

Since S has cyclic stabilizer $\pi_\Gamma(S) \subset \mathcal{D}(\Gamma)$. We claim that $\pi_\Gamma(S)$ is either a single Type 1 vertex or is the star of a Type 2 vertex. Then g fixes the appropriate vertex.

If S is a cut point this is obvious, so suppose it is a cut pair and that the image is not contained in the closed star of some Type 2 vertex. Then there is some Type 1 vertex between the two points of $\pi_\Gamma(S)$. This corresponds to a cut set S_i separating the two points of S . But this would mean S and S_i cross, contradicting the hypothesis that the S_i are uncrossed. \square

For the remainder of this section we assume the hypotheses and notation of Lemma 3.22:

- (1) $\{S_i\}_{i \in I}$ is an uncrossed collection.
- (2) Γ is the corresponding graph of groups decomposition of F .

Lemma 3.23. *If S is a cut point or cut pair of $\mathbf{D}_{\underline{w}}$ then $\alpha_\Gamma(S)$ is a cut point or cut pair of $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$. If R is a cut point or cut pair of $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$ then $\alpha_\Gamma^{-1}(R)$ is a cut point or cut pair of $\mathbf{D}_{\underline{w}}$.*

Proof. Consider an element h of $\text{Aug}_\Gamma(\underline{w}) \setminus \underline{w}$. It generates the stabilizer of an S_i that is an uncrossed cut pair in $\mathbf{D}_{\underline{w}}$. Thus, both \bar{h}^∞ and h^∞ are in the same complementary component of S . Identifying them does not reduce the number of components, so $\alpha_\Gamma(S)$ is a cut set in $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$.

Conversely, let R be a cut point or cut pair of $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$. It is clear that $\alpha_\Gamma^{-1}(R)$ is a cut set in $\mathbf{D}_{\underline{w}}$; we just need to show that it consists of at most two points.

Suppose a point $r \in R$ has preimage $\alpha_\Gamma^{-1}(r)$ consisting of two points. Then r is stabilized by a conjugate of an element $h \in \text{Aug}_\Gamma(\underline{w}) \setminus \underline{w}$, so $\alpha_\Gamma^{-1}(r) = S_i$ for some i . By the first part of the lemma, $\alpha_\Gamma(S_i) = \{r\}$ is a cut point, so $R = \{r\}$, and $\alpha_\Gamma^{-1}(R) = S_i$ is a cut pair. \square

Lemma 3.24. *Let G be a non-cyclic vertex group of Γ . Let g be an element of G such that $\delta_{\text{Ind}_\Gamma^G(\underline{w})}(\{\bar{g}^\infty, g^\infty\})$ is a cut point or uncrossed cut pair in $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$. Then $\delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ is a cut point or uncrossed cut pair, respectively, of $\mathbf{D}_{\underline{w}}$ that is not in the uncrossed collection.*

Proof. There is a Type 2 vertex of $\mathcal{D}(\Gamma)$ corresponding to G . To each adjacent Type 1 vertex there is an associated cut set S_j . Let J be the set of indices of these cut sets. Each S_j becomes a cut point in $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$, and each complementary component of the image of $\delta_{\text{Ind}_\Gamma^G(\underline{w})}(\{\bar{g}^\infty, g^\infty\})$ in $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ contains some of these S_j points, so $\alpha_\Gamma^{-1}(\delta_{\text{Ind}_\Gamma^G(\underline{w})}(\{\bar{g}^\infty, g^\infty\})) = \delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ is a cut point or cut pair in $\mathbf{D}_{\underline{w}}$ separating some of these S_j . By definition, no two cut sets in $\{S_j\}_{j \in J}$ are separated by a cut set in $\{S_i\}_{i \in I}$, so $\delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ is a cut set and is not in the uncrossed collection.

Since $\delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ is not one of the S_i , its cardinality is the same as that of the image in $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$. Thus, they are either both cut points or both cut pairs.

It remains only to show that if $\delta_{\text{Ind}_\Gamma^G(\underline{w})}(\{\bar{g}^\infty, g^\infty\})$ is an uncrossed cut pair then $\delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ is uncrossed.

Suppose $\delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ is crossed by some other cut pair R of $\mathbf{D}_{\underline{w}}$. If $\pi_\Gamma(R)$ is not contained in the star of the G vertex then there is an S_i separating a point of R from $\delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$. For R to cross $\delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ it would have to cross S_i , but S_i is uncrossed. This is a contradiction, so $\pi_\Gamma(R)$ is contained in the star of the G vertex. This means that $\alpha_\Gamma(R)$ and $\alpha_\Gamma(\delta_{\text{Ind}_\Gamma^G(\underline{w})}(\{\bar{g}^\infty, g^\infty\}))$ are crossing cut pairs in $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$. Since these both project to the star of the G vertex, their α_Γ images are in the copy of $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ embedded in $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$ as in Lemma 3.17. Thus we get crossing cut pairs of $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$, contrary to hypothesis. \square

Lemma 3.25 (Refinement Lemma). *Let G be a non-cyclic vertex of Γ . If $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ is neither a circle nor rigid then there is a refinement Γ' of Γ obtained by splitting G rel $\text{Ind}_\Gamma^G(\underline{w})$. The refinement Γ' is a splitting over an uncrossed collection containing $\{S_i\}_{i \in I}$.*

Proof. By Lemma 3.19, $\mathbf{D}_{\text{Ind}_\Gamma^G(\underline{w})}$ is connected. If it is not a circle and not rigid, then by Proposition 3.9 there is an element $g \in G$ such that $\delta_{\text{Ind}_\Gamma^G(\underline{w})}(\{\bar{g}^\infty, g^\infty\})$ is a cut point or uncrossed cut pair. By Lemma 3.24, $\delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ is a cut point or uncrossed cut pair of $\mathbf{D}_{\underline{w}}$. Thus, we can add the orbit of $\delta_{\underline{w}}(\{\bar{g}^\infty, g^\infty\})$ to the set $\{S_i\}_{i \in I}$ to get a larger uncrossed collection, and hence a graph of groups decomposition Γ' refining Γ . \square

3.5. Proof of the Decomposition Theorem. The splitting theorem, Theorem 3.12, suggests an inductive approach: split and then use Lemma 3.25 to refine it by splitting vertex groups further. By accessibility of splittings over \mathbb{Z} , as in Bestvina-Feighn [3], there can be only finitely many compatible splittings, so we get a finite graph of groups decomposition of F . This is fine, but we would like to know that graph of groups decomposition is canonical. In fact, we do not have to use an inductive procedure, we are set up to do the entire rJSJ in one fell swoop.

Theorem 3.26 (Relative JSJ-Decomposition Theorem). *There exists a canonical relative JSJ-decomposition (rJSJ), a graph of groups decomposition Γ of F relative to \underline{w} with cyclic edge groups, satisfying the following conditions:*

- (1) *If there is more than one vertex, the graph is bipartite. Cyclic vertex groups are adjacent only to non-cyclic vertex groups, and vice-versa. Furthermore, if G is a non-cyclic vertex group the incident edge groups map onto G -maximal cyclic subgroups of G in distinct G -conjugacy classes. Finally, the sum of the degrees of the edge inclusions at any cyclic vertex group is at least 2.*
- (2) *Γ is universal. If F splits over a cyclic subgroup relative to \underline{w} then the cyclic subgroup is conjugate into one of the vertex groups.*
- (3) *Γ is maximal. It can not be refined and still satisfy these conditions.*

Moreover, the rJSJ is characterized by splitting F over the stabilizers of cut points and uncrossed cut pairs in $\mathbf{D}_{\underline{w}}$. There are three mutually exclusive possibilities:

- (a) *(F, \underline{w}) is rigid. $\mathbf{D}_{\underline{w}}$ has no cut points or cut pairs. The rJSJ is trivial, a single vertex stabilized by F .*
- (b) *(F, \underline{w}) is a QH-surface. $\mathbf{D}_{\underline{w}}$ is a circle. The rJSJ is trivial, a single vertex stabilized by F .*
- (c) *The rJSJ is nontrivial. For every non-cyclic vertex group G the pair $(G, \text{Ind}_F^G(\underline{w}))$ is either rigid or a QH-surface.*

Consequently, if F splits over a cyclic subgroup relative to \underline{w} then the cyclic subgroup is conjugate into one of the cyclic vertices or one of the QH-surface vertices of the rJSJ.

Proof. If (F, \underline{w}) is rigid or is a sphere with three boundary components then there are no splittings rel \underline{w} . The rJSJ is trivial, and we are done.

If (F, \underline{w}) is a QH-surface other than a sphere with three boundary components then splittings rel \underline{w} come from essential, non-boundary parallel, simple closed curves on the surface. For any such curve we can find another intersecting it, giving us an incompatible splitting. Again, the rJSJ is trivial, and we are done.

If we are not in either of these cases then by Corollary 3.5 there exists a cut point or an uncrossed cut pair in $\mathbf{D}_{\underline{w}}$.

Take the collection $\{S_i\}_{i \in I}$ of all cut points and uncrossed cut pairs. By Proposition 3.8 there are only finitely many orbits of these, so this is an uncrossed collection. Apply Proposition 3.10 to get simplicial tree $\mathcal{D}(\Gamma)$ with a cocompact F action. This gives us a graph of groups decomposition Γ with cyclic edge groups.

The tree $\mathcal{D}(\Gamma)$ is canonically defined by the topology of the decomposition space, so the resulting graph of groups decomposition is canonical. Specifically, the conjugacy classes of vertex and edge stabilizers are determined, and the groups themselves are determined by choosing a maximal tree in the quotient graph and choosing a lift of that tree to $\mathcal{D}(\Gamma)$.

We will show that this Γ satisfies conditions (1)-(3). Conversely, we will show that any graph of groups Γ' satisfying conditions (1)-(3) has Bass-Serre tree $\mathcal{D}(\Gamma')$ equivariantly isomorphic to $\mathcal{D}(\Gamma)$, so Γ and Γ' are equivalent graph of groups decompositions.

Condition (1) says that the graph of groups is normalized as in Section 2.5. Using the facts that F is free and that the cyclic vertex groups of Γ are maximal cyclic subgroups of F it is easy to see that Γ satisfies these conditions.

Uncrossed splittings are universal by Lemma 3.22, so Γ satisfies condition (2).

If for some non-cyclic vertex G of Γ the pair $(G, \text{Ind}_\Gamma^G(\underline{w}))$ is neither rigid nor a QH-surface then by Lemma 3.25 there is a refinement of Γ coming from a larger uncrossed collection. This is absurd, we have already included all cut points and uncrossed cut pairs in our uncrossed collection. Thus, Γ satisfies condition (c).

Conditions (2) and (c) imply that any refinement of Γ would come from splitting a QH-surface vertex group. The resulting splitting would not be universal, because if there is a way to split the surface then there is always an incompatible way to split it. Thus, conditions (2) and (c) imply (3), so Γ satisfies the rJSJ conditions.

Now suppose Γ' is another graph of groups decomposition of $F \text{ rel } \underline{w}$ satisfying conditions (1)-(3). Condition (c) must also be satisfied or else it would be possible to refine Γ' in a universal way, contradicting maximality.

Consider a cyclic vertex group $\langle g \rangle$ of Γ' . We would like to show $S = \delta(\{\bar{g}^\infty, g^\infty\})$ is a cut point or uncrossed cut pair in $\mathbf{D}_{\underline{w}}$. The number of components of $\mathbf{D}_{\underline{w}} \setminus S$ is equal to the sum of the degrees of the edge maps into the vertex group. By condition (1), this is at least two, so S is a cut point if $g \in \underline{w}$ or cut pair otherwise.

If the sum of the degrees of the edge maps is greater than two then S is an uncrossed cut pair.

If the sum of the degrees is equal to two and one of the adjacent non-cyclic vertices is rigid, then S is an uncrossed cut pair, by Lemma 3.20.

Otherwise, either the vertex separates two QH-surfaces glued along boundary curves or it is adjacent to one QH-surface and the edge maps into the cyclic vertex group with degree 2.

In the first case, the cyclic vertex can be removed by gluing together the two QH-surfaces to give a larger QH-surface. In the second case, the cyclic vertex can be removed by gluing a Möbius strip to the corresponding QH-surface along their boundary curves. In either case, the new surface contains a non-boundary parallel essential simple closed curve that intersects the curve we just glued along. This would provide a splitting of $F \text{ rel } \underline{w}$ incompatible with Γ' , contradicting universality. Thus, each cyclic vertex group of Γ' is the stabilizer of a cut point or uncrossed cut pair. Furthermore, the cyclic vertex groups account for all the cut points and uncrossed cut pairs, since Γ' is maximal.

$\mathcal{D}(\Gamma)$ and $\mathcal{D}(\Gamma')$ are both equivariantly isomorphic to the tree constructed in Proposition 3.10, hence, to each other. \square

Remark. Note that if Γ is the rJSJ for a multiword \underline{w} then Γ is also the rJSJ for the augmented multiword $\text{Aug}_\Gamma(\underline{w})$. The change in passing from \underline{w} to $\text{Aug}_\Gamma(\underline{w})$ is that uncrossed cut pairs in $\mathbf{D}_{\underline{w}}$ all become cut points in $\mathbf{D}_{\text{Aug}_\Gamma(\underline{w})}$. This formalizes the claim in the introduction that there are some “missing curves” that really ought to have been included in the multiword. The missing curves correspond to uncrossed cut pairs, and including them does not change the rJSJ.

In practice one might be interested in deciding whether a given graph of groups decomposition of $F \text{ rel } \underline{w}$ is the rJSJ or not.

Corollary 3.27. *Let \underline{w} be a multiword in a free group F such that F does not split freely relative to \underline{w} . Let Γ be a graph of groups decomposition of $F \text{ rel } \underline{w}$ with cyclic edge groups satisfying the normalization hypotheses of condition (1) of Theorem 3.26. Suppose that for every non-cyclic vertex group G the pair $(G, \text{Ind}_\Gamma^G(\underline{w}))$ is either a QH-surface or rigid. Then Γ is the rJSJ unless there is a cyclic vertex*

that is adjacent only to QH-surfaces and such that the sum of the degrees of the edge inclusions is exactly two.

Example 3.28. Consider the empty multiword in the free group $F = \langle a, b \rangle$. Consider the splitting

$$\langle a, b \rangle *_Z \langle c \rangle$$

obtained by identifying $c = ab\bar{a}\bar{b}$. This is not an rJSJ. That would be absurd. All of our results depend on the standing assumption that the decomposition space is connected, but here it is just ∂F .

Example 3.29. Consider the multiword $\underline{w} = \{b, ba^2\}$ in $F_2 = \langle a, b \rangle$. Here is a splitting of $F \text{ rel } \underline{w}$ (edge maps are the obvious inclusions):

$$\langle b, a^2 \rangle *_{\langle a^2 \rangle} \langle a \rangle$$

The degree at the cyclic vertex is 2, and $a \notin \underline{w}$, so $\delta(\{\bar{a}^\infty, a^\infty\})$ is a cut pair. The complement has two components that are exchanged by the a -action. However, $G = \langle b, a^2 \rangle$ has $\text{Ind}_\Gamma^G(\underline{w}) = \{b, ba^2, a^2\}$, so $(G, \text{Ind}_\Gamma^G(\underline{w}))$ is a QH-surface, a sphere with three boundary components.

That means this splitting is not the rJSJ. Indeed, here is another splitting of $F \text{ rel } \underline{w}$ that is incompatible:

$$\langle b, (ba)^2 \rangle *_{\langle (ba)^2 \rangle} \langle ba \rangle$$

It is a splitting rel \underline{w} since b and $a(ba^2)\bar{a} = \bar{b}(ba)^2$ are in $\langle b, (ba)^2 \rangle$.

The two splittings are incompatible because $\langle a \rangle$ is not elliptic in the second splitting, while $\langle ba \rangle$ is not elliptic in the first. $\delta(\{\bar{a}^\infty, a^\infty\})$ and $\delta(\{\bar{ba}^\infty, (ba)^\infty\})$ are crossing cut pairs in $\mathbf{D}_{\underline{w}}$. In fact, $\mathbf{D}_{\underline{w}}$ is a circle. (F, \underline{w}) is a projective plane with two boundary components. The rJSJ is trivial.

If, on the other hand, the multiword were $\underline{w} = \{b, ba^2, a\}$ then $\delta(\{\bar{a}^\infty, a^\infty\})$ becomes a cut point instead of a cut pair. The second splitting is no longer a splitting rel \underline{w} . The first splitting is the rJSJ.

3.6. Quasi-isometry invariance of the rJSJ. Let (F, \underline{w}) and (F', \underline{w}') be two free groups with multiwords. Suppose there is a quasi-isometry $\phi: F \rightarrow F'$ that bijectively matches up the line patterns $\mathcal{L}_{\underline{w}}$ and $\mathcal{L}_{\underline{w}'}$ in the sense that there exists some constant C such that for every line $l \in \mathcal{L}_{\underline{w}}$ there is a line of $\mathcal{L}_{\underline{w}'}$ within Hausdorff distance C of $\phi(l)$, and vice versa. We say that ϕ *preserves line patterns*.

Such a quasi-isometry gives a homeomorphism of the corresponding decomposition spaces. In particular, it preserves the cut set structure. Thus, the Bass-Serre trees $\mathcal{D}(\Gamma)$ and $\mathcal{D}(\Gamma')$ for the respective rJSJ's of \underline{w} and \underline{w}' are the same. The trees are isomorphic, and even the kinds of vertices — cut point, uncrossed cut pair, rigid, QH-surface — are preserved. There is no reason in general that this isomorphism should be compatible with the group actions, but it will be, for instance, in the case of lifting a multiword to a finite index subgroup of F . Such a lift has rJSJ a finite sheeted covering graph of groups of the original rJSJ.

A result of Behrstock and Neumann [1] shows that there is a line pattern preserving quasi-isometry between any two QH-surface multiwords. In particular, the sphere with three boundary components is quasi-isometric to the other QH-surfaces. This is part of the motivation for insisting that the sphere with three boundary components is not rigid, even though it has no relative splittings. This example also shows that relative splittings need not be preserved by pattern preserving quasi-isometries, only universal splittings need be.

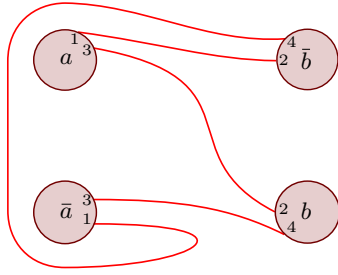
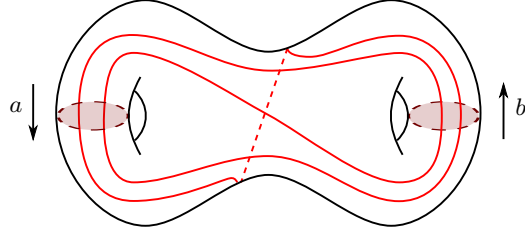
4. VIRTUALLY GEOMETRIC MULTIWORDS

A *handlebody* is a 3-manifold obtained by gluing 1-handles to a 3-ball. These are commonly imagined as thickened graphs, although we will not assume orientability.

A multiword $\underline{w} = \{w_1, \dots, w_k\}$ in $F = F_n$ is *geometric* if there exists a handlebody H with fundamental group F such that the conjugacy classes of the w_i can be represented by an embedded multicurve in the boundary of H . The multiword is *virtually geometric* if it becomes geometric upon passing to a finite index subgroup of F .

Example 4.1. $\underline{w} = \{\bar{a}\bar{b}ab\}$ in $F_2 = \langle a, b \rangle$

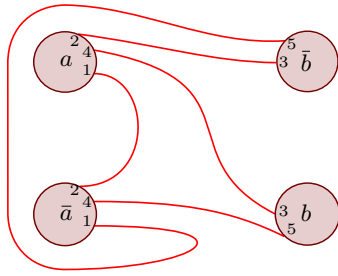
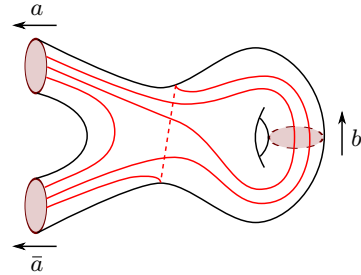
A Whitehead graph for \underline{w} is shown in Figure 11, with vertices blown up into discs. Following the edges according to the numbering reads off the word $\bar{a}\bar{b}ab$.

FIGURE 11. $\text{Wh}(\ast)\{\bar{a}\bar{b}ab\}$ FIGURE 12. Corresponding handlebody for $\bar{a}\bar{b}ab$

Embed this graph on the surface of a three ball and identify each generator disc with the disc of the inverse to get the handlebody with embedded curve shown in Figure 12.

Similarly, any multiword whose minimal Whitehead graph is a circle is geometric.

Example 4.2. $\underline{w} = \{\bar{a}^2\bar{b}ab\}$ in $F_2 = \langle a, b \rangle$

FIGURE 13. $\text{Wh}(\ast)\{\bar{a}^2\bar{b}ab\}$ FIGURE 14. Corresponding handlebody for $\bar{a}^2\bar{b}ab$

Again, we have in Figure 13 a Whitehead graph with vertices blown up to discs and a numbering around each vertex. This time, however the numbering around both the a and \bar{a} discs is $1 - 4 - 2$ going counterclockwise. We can still glue these discs together and match the numbering, but we must do it in a non-orientable way. In drawing the corresponding handlebody in Figure 14 we leave the a and

\bar{a} discs apart, but one should imagine that they have been identified to create a non-orientable a -handle.

Similarly, any multiword with a Whitehead graph that is planar and valence at most three is geometric.

We formalize the preceding examples in the following proposition:

Proposition 4.3. *If there exists a Whitehead graph of \underline{w} with a planar embedding such that the cyclic orderings of edges incident to inverse vertices are consistent, then \underline{w} is geometric.*

Proof. Embed this Whitehead graph on the surface of a 3-ball. Blow up the vertices to discs, keeping the cyclic orientation of the edges intact. Identify inverse discs, preserving the ordering of incident edges. (Equivalently, attach 1-handles joining inverse disc pairs and continue the edges across the handles.) The result is a (possibly non-orientable) handlebody with embedded multicurve representing \underline{w} . \square

In fact, more than this is true. If a multiword is geometric then every minimal Whitehead graph of \underline{w} has a planar embedding such that the cyclic orderings of edges incident to inverse vertices are consistent.

Thus, there is an algorithm to determine geometricity. The Whitehead graph is finite, so it is possible to check every distinct embedding to see if there exists one that is planar and respects cyclic ordering around the vertices.

These claims follow from work of Zieschang [8], who gives a geometric version of Whitehead's Algorithm [20] (see also Berge's *Documentation for the program Heegaard* [2]).

Example 4.4. $\underline{w} = \{\bar{a}^3 \bar{b} ab\}$ in $F_2 = \langle a, b \rangle$

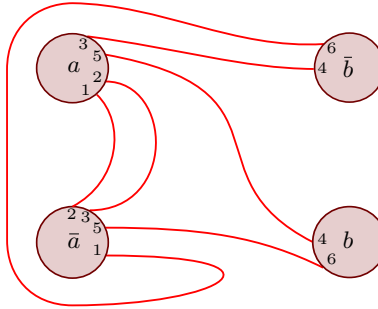


FIGURE 15. $Wh(*)\{\bar{a}^3 \bar{b} ab\}$

In Figure 15 we have a planar Whitehead graph for $\bar{a}^3 \bar{b} ab$, but the cyclic orientations around the a and \bar{a} vertices are not consistent. This Whitehead graph is minimal, and, in fact, there is no planar embedding with consistent orientations. The word $\bar{a}^3 \bar{b} ab$ is not geometric. However, we will see in Section 4.3.2 that all of the Baumslag-Solitar words $\bar{a}^q \bar{b} a^p b$ are virtually geometric.

In contrast to geometricity, until recently the only method of checking virtual geometricity was to enumerate finite index subgroups of F and check whether the multiword becomes geometric, hoping for success. Gordon and Wilton [9] asked

whether every one element multiword is virtually geometric. Manning [12] answered in the negative by constructing a class of examples for which it could be shown that every finite index subgroup has non-planar Whitehead graph.

Otal [15] did not consider the question of virtual geometricity. We will see in Section 4.1 that it follows fairly easily from his work that a rigid multiword is virtually geometric if and only if it is geometric.

Otal notes that a free splitting of the free group corresponds to a connected sum of the corresponding handlebodies, so, as usual, we will confine our attention to the case that F does not split freely rel \underline{w} .

4.1. Rigid Multiwords and Geometricity.

Lemma 4.5 ([15, Proposition 0]). *The decomposition space of a geometric multiword is planar.*

Proof. By definition, a geometric multiword can be realized by an embedded multicurve on the surface of a handlebody. The multicurve lifts to a collection of disjoint curves on the boundary surface of the universal cover of the handlebody. This universal cover is a thickened tree, and may be compactified by including the Cantor set boundary of the tree. The resulting space is a 3-ball with a collection of disjoint arcs in the bounding 2-sphere. By Moore's Decomposition Theorem [14], the quotient of the 2-sphere obtained by collapsing each of the curves to a point is still a 2-sphere. The image of the Cantor set in this quotient is exactly the decomposition space. Thus, the decomposition space embeds into S^2 . \square

Passing to a finite index subgroup induces a homeomorphism of decomposition spaces, so we also have:

Corollary 4.6. *The decomposition space of a virtually geometric multiword is planar.*

Theorem 4.7 (cf [15, Theorem 1]). *Let (F, \underline{w}) be rigid. The following are equivalent:*

- (1) *The multiword \underline{w} is geometric.*
- (2) *The decomposition space $D_{\underline{w}}$ is planar.*
- (3) *Every minimal Whitehead graph for \underline{w} has an embedding in the plane with consistent cyclic orderings of edges incident to inverse vertices.*

Proof. Lemma 4.5 shows (1) implies (2).

(2) implies (3) is the content of [15, Lemma 4.4]. The hypotheses for this lemma are that every element of the multiword is indecomposable and that the decomposition space embeds into S^2 in such a way that closures of the complementary regions intersect pairwise in at most one point. The first hypothesis is too strong. Indecomposability of each element of the multiword is only used to prove that the decomposition space has no cut points. The second hypothesis is satisfied if the decomposition space has no cut pairs. Therefore, rigidity is a sufficient hypothesis.

(3) implies (1) by Proposition 4.3. \square

Corollary 4.8. *A rigid multiword is virtually geometric if and only if it is geometric.*

Manning has shown [12] that the word $w = bbaaccabc$ in $F_3 = \langle a, b, c \rangle$ is not virtually geometric. This is proved by showing that the structure of the Whitehead graph of this word implies that the Whitehead graph in any finite cover is still non-planar.

Alternatively, using the methods of [7] it is possible to show (F, w) is rigid, so non-planarity of the Whitehead graph immediately implies the word is not virtually geometric.

Similarly, we know by Proposition 2.10 that a multiword in F_n whose Whitehead graph is the complete graph on the $2n$ vertices is rigid. Such a Whitehead graph is planar if and only if $n = 2$. When $n = 2$ the valence is 3, and any such multiword is geometric. For any $n > 2$ such a multiword is not virtually geometric.

4.2. Non-rigid Multiwords and Virtual Geometricity. In this section we prove the Characterization of Virtual Geometricity, that virtually geometric multiwords are those that are made up of geometric pieces.

We first prove the theorem in the special case that the decomposition space has no uncrossed cut pairs, see Lemma 4.9. This is the case where there are no “missing curves”, or equivalently, where they have been added to the multiword.

Lemma 4.9 contains the interesting features of the proof and avoids some technical complications. Given Lemma 4.9, the proof of Theorem 4.10 amounts to showing that adding in the “missing curves” does not change planarity of the decomposition space.

Lemma 4.9. *Let \underline{w} be a multiword in F such that the decomposition space $\mathbf{D}_{\underline{w}}$ is connected with no uncrossed cut pairs. Let Γ be the $rJSJ$. The following are equivalent:*

- (1) *The multiword \underline{w} is virtually geometric.*
- (2) *The decomposition space $\mathbf{D}_{\underline{w}}$ is planar.*
- (3) *For every non-cyclic vertex group G of the Γ , the induced multiword $\text{Ind}_{\Gamma}^G(\underline{w})$ is geometric.*

Proof. Corollary 4.6 shows (1) implies (2).

Let G be a non-cyclic vertex group of Γ . Since there are no uncrossed cut pairs, $\underline{w} = \text{Aug}_{\Gamma}(\underline{w})$, so Lemma 3.17 shows that the decomposition space $\mathbf{D}_{\text{Ind}_{\Gamma}^G(\underline{w})}$ embeds into $\mathbf{D}_{\underline{w}}$. Thus, if $\mathbf{D}_{\text{Ind}_{\Gamma}^G(\underline{w})}$ is non-planar then $\mathbf{D}_{\underline{w}}$ is non-planar as well. Therefore, (2) implies that $\mathbf{D}_{\text{Ind}_{\Gamma}^G(\underline{w})}$ is planar. If $(G, \mathbf{D}_{\text{Ind}_{\Gamma}^G(\underline{w})})$ is a QH-surface then $\text{Ind}_{\Gamma}^G(\underline{w})$ is geometric. If $(G, \mathbf{D}_{\text{Ind}_{\Gamma}^G(\underline{w})})$ is rigid then since it is planar Theorem 4.7 says it is geometric. Thus, (2) implies (3).

Now assume (3). From a graph of groups we may build a corresponding graph of spaces [17]. For each vertex group choose a vertex space with fundamental group isomorphic to the vertex group. For each edge group choose a space with fundamental group isomorphic to the edge group, and let the edge space be the product of that space with the unit interval. Use the edge injections of the graph of groups to define attaching maps of edge spaces to the corresponding vertex spaces. The resulting space will have fundamental group isomorphic to the fundamental group of the graph of groups.

For each non-cyclic vertex group, the induced multiword is geometric, so we can choose the vertex space to be a handlebody with an embedded multicurve in the boundary representing the induced multiword.

For the edge spaces we could use annuli, but later we will want to thicken them to make the resulting graph of spaces a 3-manifold.

For the moment we will also make a geometricity assumption on the cyclic vertex groups. Suppose for a cyclic vertex group $\langle g \rangle$ there are k incident edges and each edge injection is degree one. In this case we choose the vertex space to be a solid torus with $k + 1$ disjoint curves on the boundary representing the element g and k attaching curves to which we will glue a boundary curve of an annulus edge space.

Another possibility is that the degrees of the edge injections are all two except for possibly one of degree one. In this case we choose the vertex space to be a solid Klein bottle, and again we have disjoint curves on the boundary representing g and the attaching curves.

Suppose one of these possibilities is true for every cyclic vertex group.

The resulting graph of spaces has fundamental group F and has an embedded multicurve representing \underline{w} such that the multicurve is disjoint from the edge spaces. It is not yet a 3-manifold with boundary; we need to fatten the annuli. To see if this is possible, consider for each boundary component of each annulus the tubular neighborhood of the attaching curve in the boundary of the corresponding handlebody. If for each annulus the two neighborhoods are either both annuli or both Möbius strips then the annuli may be fattened to make the graph of spaces a 3-manifold. Now, a fattened annulus is composed of a 1-handle and a 2-handle, so this does not explicitly give the resulting space a handlebody structure. However, a graph of aspherical spaces is aspherical [17], and a compact aspherical 3-manifold with free fundamental group is a handlebody [11], so \underline{w} is geometric.

Thus, assuming (3), there are two possible obstructions to geometricity:

- (1) The degrees of the edge injections into some cyclic vertex group are not of one of the two forms described above.
- (2) Some annulus can not be fattened because one boundary neighborhood is an annulus and the other is a Möbius strip.

Claim 4.9.1. These obstructions vanish in a finite index subgroup of F , so \underline{w} is virtually geometric.

Proof of Claim. There are finitely many elements $g_i \in \underline{w}$ such that $\delta(\{\bar{g}_i^\infty, g_i^\infty\})$ is a cut point in $\mathbf{D}_{\underline{w}}$.

From the proof of Proposition 3.11, an edge injection of degree greater than one into a cyclic vertex group $\langle g_i \rangle$ occurs when the g_i -action permutes some components of $\mathbf{D}_{\underline{w}} \setminus \delta(\{\bar{g}_i^\infty, g_i^\infty\})$. There are only finitely many components, so there exists some minimal positive power a_i of g_i such that the $g_i^{a_i}$ -action fixes each complementary component.

Additionally, if some edge incident to the $\langle g_i \rangle$ vertex attaches to a handlebody around a non-orientable handle, and if a_i is odd, then consider $g_i^{2a_i}$.

Let H_i be a finite index subgroup of F in which $g_i^{a_i}$ (or $g_i^{2a_i}$) is basic. Let H be the finite index subgroup $\cap_i H_i$. If we apply the Relative JSJ-Decomposition Theorem to H we get a graph of groups covering the graph of groups decomposition for F . By construction, the smallest power of g_i in H is a multiple of $g_i^{a_i}$, so all edge inclusions are degree one. This takes care of obstruction (1), and we can choose all the cyclic vertex spaces to be solid tori.

Furthermore, we can take the vertex spaces to be handlebodies finitely covering the original handlebodies. If some attaching curve in the original decomposition

ran along a Möbius strip then it runs along an even covering of the Möbius strip in the covering handlebodies. Thus, all attaching curves have annulus neighborhoods, which takes care of obstruction (2). \diamond

Thus, (3) \implies (1). \square

Theorem 4.10 (Characterization of Virtual Geometricity). *The following are equivalent:*

- (1) *The multiword is virtually geometric.*
- (2) *The decomposition space is planar.*
- (3) *For every non-cyclic vertex group of the rJSJ, the induced multiword is geometric.*

Thus, virtually geometric multiwords are exactly those that are built from geometric pieces.

Proof. Corollary 4.6 shows (1) implies (2).

Claim 4.10.1. If $\mathbf{D}_{\underline{w}}$ is planar then for each non-cyclic vertex group G of the rJSJ, the decomposition space $\mathbf{D}_{\text{Ind}_F^G(\underline{w})}$ is planar.

Proof of Claim. For each cut point $p \in \mathbf{D}_{\underline{w}}$ there is a unique component \mathcal{C}_p of $\mathbf{D}_{\underline{w}} \setminus p$ such that the image of ∂G is contained in $\mathcal{C}_p \cup p$. The intersection over all cut points of the sets $\mathcal{C}_p \cup p$ is a connected subset of $\mathbf{D}_{\underline{w}}$ containing the image of ∂G . Thus, we may assume that there are no cut points.

Now, let $\{g_1, \dots, g_k\} = \text{Ind}_F^G(\underline{w}) \setminus \underline{w}$, so that $\{\bar{g}_i^\infty, g_i^\infty\}$ gives an uncrossed cut pair in $\mathbf{D}_{\underline{w}}$. This means that $\mathbf{D}_{\text{Ind}_F^G(\underline{w})}$ is a quotient of the image of ∂G in $\mathbf{D}_{\underline{w}}$ obtained by identifying pairs of points $\{h\bar{g}_i^\infty, hg_i^\infty\}$ for each $h \in G$ and $i = 1, \dots, k$.

Embed $\mathbf{D}_{\underline{w}}$ into S^2 . We will show that there is a monotone upper semi-continuous decomposition of S^2 whose non-degenerate elements are arcs whose two endpoints are $\{h\bar{g}_i^\infty, hg_i^\infty\}$ for some h and i , and whose interiors are disjoint from the image of ∂G in S^2 . Moore's Decomposition Theorem [14] says that the quotient of the sphere obtained by collapsing each of these arcs to a point is again the sphere, and the image of ∂G in this quotient is $\mathbf{D}_{\text{Ind}_F^G(\underline{w})}$. Thus, $\mathbf{D}_{\text{Ind}_F^G(\underline{w})}$ is planar.

A *decomposition* of S^2 is just a way of writing S^2 as a disjoint union of finite unions of compact continua. The *non-degenerate* elements are the non-singletons. The decomposition is monotone if each element is connected. The collection is upper semi-continuous if for each element A of the decomposition, and for each neighborhood U of A , there exists a neighborhood V of A such that any element of the decomposition that meets V is contained in U . For the quotient to be the sphere we also require that the elements of the decomposition are non-separating subsets.

Each uncrossed cut pair $\delta_{\underline{w}}(\{h\bar{g}_i^\infty, hg_i^\infty\})$ has finitely many complementary components in $\mathbf{D}_{\underline{w}}$, one of which, \mathcal{C} , contains the image of ∂G . Choose a connected component of $S^2 \setminus \mathbf{D}_{\underline{w}}$ that limits to $\delta_{\underline{w}}(h\bar{g}_i^\infty)$ and $\delta_{\underline{w}}(hg_i^\infty)$. The boundary of this set is a Jordan curve passing through the two points $\delta_{\underline{w}}(h\bar{g}_i^\infty)$ and $\delta_{\underline{w}}(hg_i^\infty)$.

Choose the arc connecting $\delta_{\underline{w}}(h\bar{g}_i^\infty)$ and $\delta_{\underline{w}}(hg_i^\infty)$ in our decomposition to be the sub-arc of this curve that does not go through the component of $\mathbf{D}_{\underline{w}} \setminus \{\delta_{\underline{w}}(h\bar{g}_i^\infty), \delta_{\underline{w}}(hg_i^\infty)\}$ containing the image of ∂G .

To satisfy the requirements of Moore's theorem, we must show that for every arc A and every neighborhood U of A there is a neighborhood V of A such that if an arc A' meets V it is contained in U .

By construction, each interior point of each arc has a neighborhood that is not entered by any other arc in the collection, but this is not true for the endpoints of the arc.

Fix an arc A and let b be one of its endpoints. Let \hat{b} be the point of $\partial\mathcal{T}$ that is the preimage of this endpoint.

The identity element of F gives a basepoint $*$ for the topology on $\overline{\mathcal{T}}$ in the sense that the basic open neighborhoods $N_r(\xi)$ of a point $\xi \in \partial\mathcal{T}$ are the subsets of $\overline{\mathcal{T}}$ consisting of points η such that the geodesic from $*$ to η coincides with the geodesic from $*$ to ξ for at least distance r . Let R be the length of the longest subword common to two elements of $\text{Aug}_\Gamma(\underline{w})$ (as cyclic words). In other words, R is the maximum overlap of two different axes in the line pattern. If l^+ and l^- are the two endpoints of a line l in the line pattern, and if one endpoint of another line l' is in $N_{r+R}(l^+)$ for some $r > d_\mathcal{T}(*, l)$, then the other endpoint of l' is in $N_r l^+$. Furthermore, for every $r > d_\mathcal{T}(*, l)$ it is also true that $\delta_{\underline{w}}^{-1}(\delta_{\underline{w}}(N_{r+R}(l^+))) \subset N_r(l^+) \cup l^-$.

Thus, for any sufficiently large r there is an open neighborhood U_r of b in $\mathbf{D}_{\underline{w}}$ such that $\delta_{\underline{w}}(N_{r+R}(\hat{b})) \subset U_r \subset \delta_{\underline{w}}(N_r(\hat{b}))$.

Let U be any neighborhood of A . For each interior point γ of A there is a small neighborhood V_γ contained in U that does not contain points of the other arcs. If b is an endpoint of A there is a small open neighborhood of b contained in U of the form U_r for some r , as in the previous paragraph. Let V_b be a neighborhood of b of the form U_{r+2R} . Let V be the union of the U_a for all $a \in A$. If any arc of the collection enters V then it must enter $V_b \subset U_{r+2R}$. This implies that the entire arc is contained in $\delta_{\underline{w}}(N_{r+R}(\hat{b})) \subset U$. \diamond

Claim 4.10.1, Theorem 4.7 and the fact that QH-surface multiwords are always geometric show that (2) implies (3).

By Lemma 4.9, if the induced multiword in each non-cyclic vertex group of the rJSJ is geometric in its vertex group then $\text{Aug}_\Gamma(\underline{w})$ is virtually geometric. Clearly this implies \underline{w} is virtually geometric; it is a subset of $\text{Aug}_\Gamma(\underline{w})$. Just take the graph of spaces from Lemma 4.9 and omit some of the curves running around solid torus vertex spaces to get an embedded multicurve representing \underline{w} . Thus (3) implies (1). \square

Remark. An argument similar to that of Claim 4.10.1 may be used to see Theorem 3.18 from a different viewpoint, at least for virtually geometric multiwords. Let \underline{w} be a multiword with connected, planar decomposition space. Suppose there is an uncrossed cut pair with k complementary components. It is possible to embed a graph in S^2 with the two points as vertices and k edges such that the interiors of the edges lie in the complement of the decomposition space, and such that complementary components of the decomposition space lie in complementary components of the sphere minus the graph. We can find a similar graph for all uncrossed cut pairs and cut points, and make the collection upper semi-continuous.

In general, the quotient of the sphere obtained by collapsing all of these graphs to points is a *cactoid*, but in this case we can say more: it is a tree of spheres whose tree structure mirrors that of the Bass-Serre tree of the rJSJ. The vertices of the

Bass-Serre tree with non-cyclic stabilizers correspond to spheres. The cyclic vertices correspond to points of intersection of different spheres. The decomposition space of the augmented multiword embeds into the tree of spheres in such a way that the decomposition space of an induced multiword in a non-cyclic vertex group of the rJSJ embeds into the appropriate sphere in the tree of spheres, and the cut points of the decomposition space are exactly points of intersection of multiple spheres.

4.3. Examples.

4.3.1. *Baumslag's Word*. The first example is *Baumslag's word* $w = \bar{a}^2 \bar{b} a b \bar{a} b$ in $F_2 = \langle a, b \rangle$. In response to a question of Gordon and Wilton, Manning showed, by enumerating subgroups and checking geometricity, that this word becomes geometric in an orientable handlebody with fundamental group an index four subgroup of F_2 .

The rJSJ for $F = \langle a, b \rangle \cong \langle a, b, c \mid c = \bar{b}ab \rangle$ is shown in Figure 16.

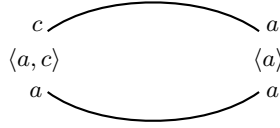


FIGURE 16. rJSJ-Decomposition of $\langle a, b \rangle$ for $\bar{a}^2 \bar{b} a b \bar{a} b$ ($c = \bar{b}ab$)

The word w becomes $\bar{a}^2 \bar{c} a c$ when rewritten in the rank two vertex group, so the induced multiword is $\{\bar{a}^2 \bar{c} a c, a, c\}$. One can check that this multiword is rigid, so this is the rJSJ. (Unfortunately the Whitehead graph contains two edges joining \bar{a} to a , so Proposition 2.10 does not apply here. Checking rigidity takes a little work.)

In Figure 17 we have a reduced Whitehead graph/Heegaard diagram for induced multiword that shows it is geometric. Figure 18 shows a (non-orientable) handlebody with embedded multicurve representing $\{\bar{a}^2 \bar{c} a c, a, c\}$.

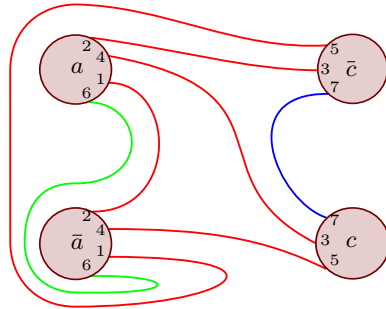


FIGURE 17. Whitehead graph/Heegaard diagram

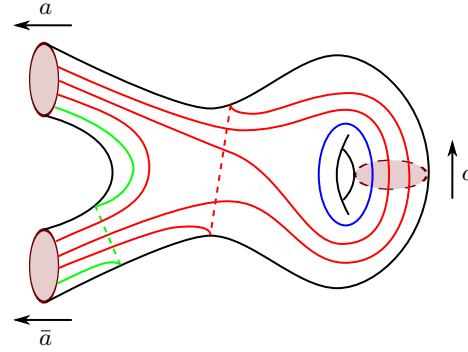


FIGURE 18. Corresponding non-orientable handlebody for $\{\bar{a}^2 \bar{c} a c, a, c\}$

The obstruction to geometricity of w is that if we tried to build a graph of spaces with this non-orientable handlebody as the vertex space we would need to conjugate a curve running around an orientable handle to one running around a

non-orientable handle (a is conjugate to c). A neighborhood of the a curve is a Möbius strip, while a neighborhood of the c curve is an annulus, so we can not achieve this conjugation with a fattened annulus.

To correct this problem, pass to the index two subgroup:

$$G = \langle A = a^2, b, B = ab\bar{a} \rangle$$

Baumslag's word is not in this subgroup, but its square is:

$$\begin{aligned} w^2 &= (\bar{a}^2 \bar{b} \bar{a} b \bar{a} b)^2 \\ &= \bar{a}^2 \bar{b} \bar{a} b \bar{a} b \bar{a}^2 \bar{b} \bar{a} b \bar{a} b \\ &= \bar{a}^2 \cdot \bar{b} \cdot \bar{a}^2 \cdot ab\bar{a} \cdot a^2 \cdot \bar{b} \cdot ab\bar{a} \cdot \bar{a}^2 \cdot a\bar{b}\bar{a} \cdot b \cdot a\bar{b}\bar{a} \cdot a^2 \cdot b \\ &= \bar{A} \bar{b} \bar{A} B \bar{A} \bar{b} B \bar{A} \bar{B} b \bar{B} A b \end{aligned}$$

Apply the Whitehead automorphism that pushes B through b . This sends B to bB and \bar{B} to $\bar{B}\bar{b}$, and fixes b and A . The word becomes $\bar{A}(\bar{b}\bar{A}b)BAB\bar{A}\bar{B}^2(\bar{b}Ab)$.

The splitting over $\langle A \rangle$ is therefore $\langle A, b, B, C \mid C = \bar{b}Ab \rangle$. The induced multiword in the vertex group $\langle A, B, C \rangle$ is $\{A, C, \bar{A}C BAB \bar{A} \bar{B}^2 C\}$. This multiword is rigid (this takes some work) and geometric in a non-orientable handlebody, as seen in Figure 19.

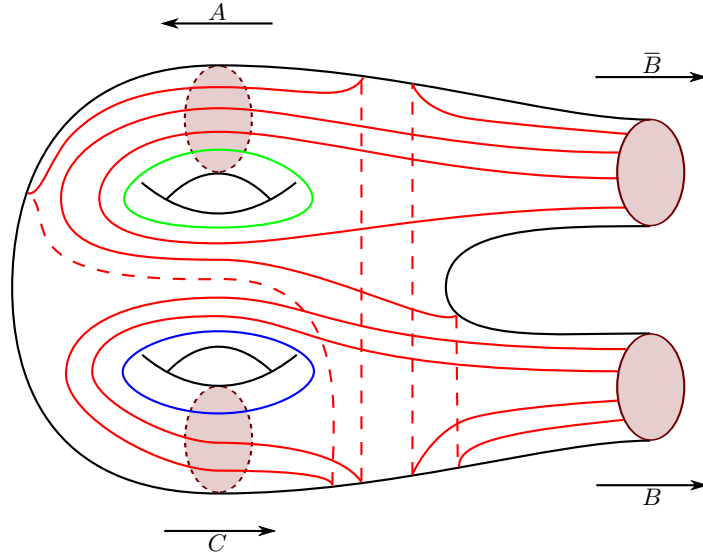


FIGURE 19. A non-orientable handlebody for $\{A, C, \bar{A}C BAB \bar{A} \bar{B}^2 C\}$

Although the handlebody is non-orientable, this time we can build a 3-manifold graph of spaces because we only need to conjugate an orientable handle to an orientable handle. Gluing on a fattened annulus conjugating A to C gives a non-orientable handlebody with fundamental group isomorphic to G for which the image of w^2 is geometric. One could pass further to a twofold cover of this handlebody to find an index four subgroup of F for which the multiword is geometric in an orientable handlebody, if desired.

4.3.2. *Baumslag-Solitar Words.* Another interesting family of examples are given by the Baumslag-Solitar words $w_{p,q} = \bar{a}^q \bar{b} a^p b$ in $F_2 = \langle a, b \rangle$. We will assume that $0 < p \leq q$. Gordon and Wilton [9] have shown that $w_{p,q}$ is virtually geometric when p and q are relatively prime.

The decomposition space associated to this word is connected without cut points. The pair $\delta(\{\bar{a}^\infty, a^\infty\})$ is a cut pair. Figure 20 shows the Whitehead graph:

$$\text{Wh}_{\{a,b\}}([*, a^6])\{\bar{a}^3 \bar{b} a^2 b\} \setminus [a^{-\infty}, a^\infty]$$

This is six copies of $\text{Wh}(\ast)$ spliced together. Note there are $p = 2$ components containing the vertices along the top of the figure, and $q = 3$ components containing the vertices along the bottom. The a -action is a shift that exchanges the two components on the top and cyclically permutes the three components along the bottom, so the a^6 -action fixes all five components.

In general there are $p+q$ connected components in the complement of $\delta(\{\bar{a}^\infty, a^\infty\})$. There are two orbits of components under the a -action, one of size p and one of size q .

The case when $p = q = 1$ is special; in this case the Whitehead graph is a circle, which implies the decomposition space is a circle and the word is geometric.

Otherwise, the number of complementary components is $p+q > 2$, so $\delta(\{\bar{a}^\infty, a^\infty\})$ is an uncrossed cut pair. The rJSJ is shown in Figure 21.

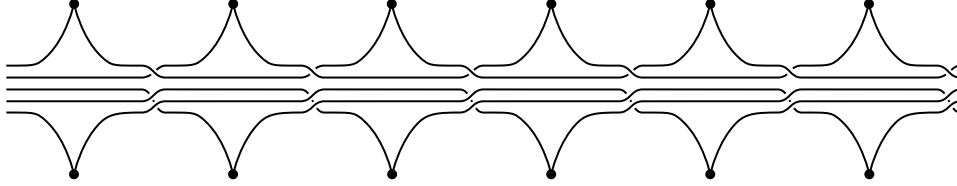


FIGURE 20. $\text{Wh}_{\{a,b\}}([*, a^6])\{\bar{a}^3 \bar{b} a^2 b\} \setminus [a^{-\infty}, a^\infty]$

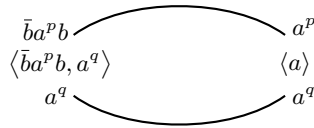


FIGURE 21. rJSJ-Decomposition of $\langle a, b \rangle$ for $\bar{a}^q \bar{b} a^p b$

The rank two vertex group is $\langle A = a^q, C = \bar{a}^q \bar{b} a^p b \rangle$, and the induced multiword in this vertex group is $\{A, C, \bar{A}C\}$. The Whitehead graph for this multiword is a circle, which implies the vertex decomposition space is a circle and the induced multiword is geometric. Thus, Theorem 4.10 says $w_{p,q}$ is at least virtually geometric.

The cyclic vertex group has edge inclusions of degrees p and q .

If $p = 1$ and $q = 2$ we can make this geometric by using a solid Klein bottle for the cyclic vertex space. (We saw the non-cyclic vertex space for this example back in Example 4.2.)

If $p = q$ the word is also geometric, because two disjoint degree p curves fit into the boundary of a solid torus. However, an additional degree one curve does not fit. This example is notable because the word $w_{p,p}$ is geometric, but the augmented

multiword $\{w_{p,p}, a\}$ is only virtually geometric. Augmenting the multiword does not change virtual geometricity, but it may change geometricity.

In all other cases, the word $w_{p,q}$ is not geometric. Let m be the least common multiple of p and q . It suffices to pass to the index m subgroup:

$$G = \langle A, B_0, B_1, \dots, B_{m-1} \mid A = a^m, B_i = a^i b \bar{a}^i \rangle$$

Since $A = a^m$ is the smallest power of a in this subgroup, there is still only one orbit of uncrossed cut pair in the decomposition space. The A -action fixes each of the $p + q$ complementary components of $\delta(\{\bar{A}^\infty, A^\infty\})$. Therefore, the rJSJ has a single cyclic vertex group with all edge inclusions of degree one, and all non-cyclic vertex groups of the rJSJ have a circle for the decomposition space of the induced multiword.

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